



Márcio da Silva Passos Telles

**Some examples of asymptotic combinatorial
behavior, zero-one and convergence results on
random hypergraphs**

Tese de Doutorado

Thesis presented to the Programa de Pós-Graduação em Matemática of the Departamento de Matemática, PUC-Rio as partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor: Prof. Nicolau Corção Saldanha

Rio de Janeiro
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Abstract

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Random graphs (and more generally hypergraphs) have been extensively studied, including their first order logic. In this thesis we focus on certain specific aspects of this vast theory. We consider the binomial model $H^{d+1}(n, p)$ of the random $(d + 1)$ -uniform hypergraph on n vertices, where each edge is present, independently of one another, with probability $p = p(n)$. We are particularly interested in the range $p(n) \sim C \log(n)/n^d$, after the double jump and near connectivity. We prove several zero-one, and, more generally, convergence results and obtain combinatorial applications of some of them.

Keywords

Random Structures; Graphs; Hypergraphs; First Order Logic; Zero-One Laws;

Resumo

Telles, Márcio da Silva Passos; Saldanha, Nicolau Corção. **Alguns exemplos de comportamento assintótico e resultados de convergência de tipo zero-um em hipergrafos aleatórios.** Rio de Janeiro, 2013. 64p. Tese de Doutorado — Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Grafos aleatórios (ou, mais geralmente, hipergrafos) têm sido estudados extensamente, inclusive sua lógica de primeira ordem. Nesta tese focamos em certos aspectos desta vasta teoria. Consideramos o modelo binomial $H_{d+1}(n, p)$ do hipergrafo $(d + 1)$ -uniforme aleatório com n vértices, onde cada aresta está presente, independentemente das demais, com probabilidade $p(n)$. Estamos particularmente interessados na faixa $p(n) \sim C \log(n)/n^d$, após o double jump e perto da conexidade. Demonstramos alguns resultados de tipo zero-um e, mais geralmente, de convergência e obtemos aplicações combinatórias de alguns deles.

Palavras-chave

Estruturas Aleatórias; Grafos; Hipergrafos; Lógica de Primeira Ordem; Leis de Zero-Um;

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1

Introduction

It has now been more than fifty years since Erdős and Rényi laid the foundations for the study of random graphs on their seminal paper *On the evolution of random graphs* [1], where they considered the binomial random graph model $G(n, p)$. This consists of a graph on n vertices where each of the potential $\binom{n}{2}$ edges is present with probability p , all these events being independent of each other. Many interesting asymptotic questions arise when n tends to ∞ and we let p depend on n .

Among other results, they showed that many properties of graphs exhibit a *threshold behavior*, meaning that the probability that the property holds on $G(n, p)$ turns from near 0 to near 1 in a narrow range of the edge probability p . More concretely, given a property P of graphs, in many cases there is a *threshold function* $p : \mathbb{N} \rightarrow [0, 1]$ such that, as $n \rightarrow \infty$, the probability that $G(n, \tilde{p})$ satisfies P tends to 0 for all $\tilde{p} \ll p$ and tends to 1 for all $\tilde{p} \gg p$. Erdős and Rényi showed, for example, that the threshold for connectivity is $p = \frac{\log n}{n}$. They also showed that there is a profound change in the component structure of $G(n, p)$ for p around $\frac{1}{n}$, where one of its many connected components suddenly becomes dramatically larger than all others, a phenomenon mainly understood today as a phase transition. The range of p where this occur is called the *Double Jump* and has received enormous attention from researchers since then.

The above threshold behaviors suggest that one could expect to describe some convergence results, where the probabilities of all properties in a certain class converge to known values as $n \rightarrow \infty$. Among the first convergence results there are the *zero-one laws*, where all properties of graphs expressible by a first order formula (called *elementary properties*) converge to 0 or to 1. This happens, for example, if p is independent of n . Many other instances of zero-one laws for random graphs were obtained by Joel Spencer in the book *The Strange Logic of Random Graphs* [2]. There he shows that zero-one laws hold if p lies between a number of “critical” functions. More concretely, if p satisfies one of the following conditions

- (a) $p \ll n^{-2}$
- (b) $n^{-\frac{1+l}{l}} \ll p \ll n^{-\frac{2+l}{1+l}}$ for some $l \in \mathbb{N}$

- (c) $n^{-1-\epsilon} \ll p \ll n^{-1}$ for all $\epsilon > 0$
- (d) $n^{-1} \ll p \ll (\log n)n^{-1}$
- (e) $(\log n)n^{-1} \ll p \ll n^{-1+\epsilon}$ for all $\epsilon > 0$

then a zero-one law holds.

Note that clause 2 is, in fact, a *scheme* of clauses. Note also that 1 can be viewed as a special case of 2. There are functions $p \ll n^{-1+\epsilon}$ not considered by any of the above conditions. Such “gaps” occur an infinite number of times near the critic functions in the scheme 2 and two more times: one between clauses 3 and 4 and the other between clauses 4 and 5. Spencer shows that, for some functions p conveniently near that “critic” functions in 2 and the critic functions n^{-1} and $\frac{\log n}{n}$, corresponding to the gaps 3 – 4 and 4 – 5 respectively, the probabilities of all elementary properties converge to constants $c \in [0, 1]$ as $n \rightarrow \infty$. This situation, more general than that of a zero-one law, is called a *convergence law*.

One sees immediately that the possibility that an edge probability function oscillates infinitely often between two different values can be an obstruction to getting convergence laws. With this difficulty in mind, we consider the edge probability functions $p : \mathbb{N} \rightarrow [0, 1]$ that belong to Hardy’s class of *logarithmo-exponential functions*. This class is entirely made of eventually monotone functions, avoiding the above mentioned problem, and has the additional convenience of being closed by elementary algebraic operations and compositions that can involve logarithms and exponentials. All thresholds of natural properties of graphs seem to belong in Hardy’s class.

Generally speaking, our work implies that, once one restricts the edge probabilities to functions in Hardy’s class, there are no further “gaps”: all logarithmo-exponential edge probabilities $p \ll n^{-1+\epsilon}$ are convergence laws. The arguments in Spencer’s book are sufficient for getting most of these convergence laws, except for those in the window $p \sim C \frac{\log n}{n}$, $C > 0$, where just the value $C = 1$ is discussed.

One of our main interests lies in the completion of the discussion of the convergence laws in the window $p \sim C \frac{\log n}{n}$ for other values of C and generalizations of the beautiful arguments in Spencer’s book to random uniform hypergraphs. We will see that this window hides an infinite collection of zero-one and convergence laws and that those can be presented in a simple organized fashion. We also get simple axiomatizations of the almost sure theories and describe all elementary types of the countable models of these theories.

Convergence laws have a deep connection with some elementary concepts of logic. More precisely, zero-one laws occur when the class T_p of almost sure elementary properties is *complete* and convergence laws occur when this “almost sure theory” is, in a sense, almost complete. Some everyday results in first-order logic imply that when all countably infinite models of T_p are *elementarily equivalent* (that is, satisfy the same elementary properties), T_p is complete. This is obviously the case if there is, apart from isomorphism, only one such model: in this case, we say that T_p is \aleph_0 -categorical. We will face situations where the countable models of T_p are, indeed, unique up to isomorphism. In other cases, the almost sure theory is still complete but the countable models are not unique: in the instances of the latter situation, the countable models are not far from being uniquely determined and, in particular, lend themselves to an exhaustive characterization. Finally, there are cases where T_p is not complete but we still have convergence laws: in these cases, the almost sure theories are not far from being complete, and we still manage to classify their countable models.

Along the way, we describe some combinatorial aspects of the component structure of the random hypergraph in the window $p \sim C \frac{\log n}{n}$, including some estimates of the size of the complement of its largest connected component. As a consequence, we get some elementary approximations of non-elementary events that work for probability edge functions in Hardy’s class. The phase transition occurring in the Double Jump $p \sim Cn^{-1}$ has recently been seen to hold, in this more general context of random $(d + 1)$ -uniform hypergraphs, in the window $p \sim Cn^{-d}$ by Schmidt-Przhan and Shamir in [3]. We do not discuss convergence laws in this window for $d > 1$ as Spencer successfully does for $d = 1$, although this seems an interesting question worthy of clarification in the future.

2 Preliminaires

2.1

The model $G^{d+1}(n, p)$

We consider the binomial model $G^{d+1}(n, p)$ of random $(d + 1)$ -uniform hypergraphs on n vertices with probability $p \in [0, 1]$, that is to say, the finite probability space on the set of all hypergraphs on n labelled vertices where each edge is a set of cardinality $d + 1$ and if H is such a hypergraph with k edges then one has

$$\mathbb{P}[H] = p^k(1 - p)^{\binom{n}{d+1} - k}.$$

Another useful characterization of the same probability space is to insist that each of the potential $\binom{n}{d+1}$ edges be present in $G^{d+1}(n, p)$ with probability p , each of these events being independent of each other. We will, more often, prefer the latter because it is more convenient in applications.

In the literature, the reader will find that the notations $G^{d+1}(n, p)$, $H^{d+1}(n, p)$ and even some others also stand for $G^{d+1}(n, p)$. Our choice reflects our mere personal taste.

Our interest lies on the asymptotic behavior of $G^{d+1}(n, p)$ when $n \rightarrow \infty$ and $p = p(n)$ is a function of n . More specifically, a *property* of hypergraphs is a class of hypergraphs closed by hypergraph isomorphism. Each property P gives rise to a sequence

$$\mathbb{P}(P) = \mathbb{P}[P](n, p) := \mathbb{P}[G^{d+1}(n, p(n)) \models P]$$

and it is the asymptotic behavior of these sequences we shall be interested in: when they converge; if so, what the limits are and so on.

A property P is said to hold *asymptotically almost surely* (or simply *almost surely*) if $\mathbb{P}[P] \rightarrow 1$. In this case we say simply that P holds *a.a.s.* A property P is said to hold *almost never* if its negation $\neg P$ holds almost surely. Very often, it is the case that a property P turns from holding almost never to holding almost surely in a narrow range of the edge probability p .

(2.1) Definition. We say $\tilde{p} : \mathbb{N} \rightarrow [0, 1]$ is a *threshold function* (or simply a *threshold*) for P if both the following conditions hold:

- (a) If $p \gg \tilde{p}$ then P holds a.a.s. in $G^{d+1}(n, p)$.
- (b) If $p \ll \tilde{p}$ then $\neg P$ holds a.a.s. in $G^{d+1}(n, p)$.

Above and in all that follows, for eventually positive functions $f, g : \mathbb{N} \rightarrow \mathbb{R}$, both expressions $f \ll g$ and $f = o(g)$ mean $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$.

Note that thresholds, when they exist, are not uniquely defined. For example, if \tilde{p} meets the requirements of being a threshold for P then all functions $c \cdot \tilde{p}$, with $c \in \mathbb{R}_+^*$, also do. Therefore, strictly speaking, it would be more correct to talk about “representatives” of the threshold. However, as distinctions of this nature usually play no role in what follows, we shall not bother the reader with them.

As far as thresholds are concerned, the following is a generalization of a classical result of Erdős, Rényi and Bollobás, stated and proved by Vantsyan in [4].

(2.2) Theorem. *Fix a finite $(d + 1)$ -uniform hypergraph H and let*

$$\rho := \max \left\{ \frac{|E(\tilde{H})|}{|V(\tilde{H})|} \mid \tilde{H} \subseteq H, E(\tilde{H}) > 0 \right\}.$$

Then the function $p(n) = n^{-\frac{1}{\rho}}$ is a threshold for the property of containment of H as a sub-hypergraph.

That is to say, $n^{-\frac{1}{\rho}}$ is a threshold for the appearance of small sub-hypergraphs with maximal *density* ρ . We will need this later.

Among similar results, we will see that $p(n) = \frac{\log n}{n^d}$ is a threshold for $G^{d+1}(n, p)$ being connected, apart from getting thresholds for other properties.

2.2

Logarithmico-Exponential Functions

Generally speaking, our results relate to convergence. That is, showing that for all functions p and properties P in certain specified classes, the limit

$$\lim_{n \rightarrow \infty} \mathbb{P}(P)(n, p)$$

exists. Moreover, one can usually get a nice description of these limits.

One obvious obstruction to getting such results is the possibility that the function p can oscillate between two different values, so that the corresponding probabilities also do. This would, obviously, rule out convergence.

To overcome this difficulty, one can restrict the possible functions p to a class entirely made of eventually monotone functions. One natural such choice is Hardy's class of *logarithmico-exponential functions*, or *L-functions* for short, consisting of the eventually defined real-valued functions defined by a finite combination of the ordinary algebraic symbols and the functional symbols $\log(\dots)$ and $\exp(\dots)$ on the variable n . To avoid trivialities such as

$$\frac{e^{\sqrt{-n^2}} - e^{-\sqrt{-n^2}}}{2}$$

we require that, in all "stages of construction", the functions take only real values.

By induction on the complexity of *L-functions*, one can easily show that this class meets our requirement and even more. We state the following and refer the interested reader to Hardy's book *Orders of Infinity* [5] for a proof.

(2.3) Theorem. *Any L-function is eventually continuous, of constant sign, and monotonic, and, as $n \rightarrow +\infty$, converges to a definite limit or tends to $\pm\infty$. In particular, if f and g are eventually positive L-functions, exactly one of the following relations holds.*

- (a) $f \ll g$
- (b) $f \gg g$
- (c) $f \sim c \cdot g$, for some constant $c \in \mathbb{R}$.

Thresholds of natural properties of graphs and hypergraphs appear to have representatives that are *L-functions*. (Stating and deciding a formal counterpart of that claim seems to be an interesting problem) This makes the choice of *L-functions* in the context of random hypergraphs a rather natural one.

2.3

First Order Logic of Hypergraphs

Having narrowed the class of possible edge probability functions, we now turn to a similar procedure on the class of properties of hypergraphs.

The *first order logic of $(d+1)$ -uniform hypergraphs* \mathcal{FO} is the relational logic with language $\{\sigma\}$, where σ is a $(d+1)$ -ary predicate. The semantics of \mathcal{FO} is given by quantification over vertices and giving the formula $\sigma(x_0, x_1, \dots, x_d)$ the interpretation " $\{x_0, x_1, \dots, x_d\}$ is an edge".

We say a property P of $(d+1)$ -uniform hypergraphs is *elementary* if it can be represented by a formula in \mathcal{FO} . In this case, we write $P \in \mathcal{FO}$ and,

when no possibility of confusion arises, make no notational distinction between P and the first order formula defining it.

A *first order theory* is simply a subclass $\mathcal{C} \subseteq \mathcal{FO}$, that is, a class of elementary properties.

Fix a first order theory $\mathcal{C} \subseteq \mathcal{FO}$ and a property $P \in \mathcal{FO}$. We say P is a *semantic consequence* of \mathcal{C} , and write $\mathcal{C} \models P$, if P is satisfied in all hypergraphs that satisfy all of \mathcal{C} , that is to say, if $H \models \mathcal{C}$ implies $H \models P$. One can define a deductive system in which all derivations are finite sequences of formulae in \mathcal{FO} , giving rise to the concept of P being a *syntactic consequence* of \mathcal{C} , meaning that some \mathcal{FO} -formula defining P is the last term of a derivation that only uses formulae in \mathcal{C} as axioms.

One piece of information in Gödel's Completeness Theorem is the fact that one can pick such a deductive system in a suitable fashion so as to make the concepts of semantic and syntactic consequences identical. As derivations are finite sequences of formulae, the following *Compactness Result* is obvious.

(2.4) Proposition. *If P is a semantic consequence of \mathcal{C} then it is a semantic consequence of a finite subclass of \mathcal{C} .*

In particular, if every finite subclass of \mathcal{C} is consistent then \mathcal{C} is consistent.

The “in particular” part comes from substitution of P by any contradictory property. A careful analysis of the argument on the proof of Gödel's Completeness Theorem shows the *Downward Löwenheim-Skolem Theorem*, that if \mathcal{C} is a consistent theory (that is, satisfied by some hypergraph) then there is a hypergraph on a countable number of vertices satisfying \mathcal{C} .

In spite of our particular interest in elementary properties, our interest is by no means exclusive. Rather we will, at times, discuss relations among elementary properties and provably non-elementary ones. As a matter of example and also for future reference, we define the events D_l .

In what follows, recall that the *incidence graph* $G(H)$ of a hypergraph H is a bipartite graph with $V(H)$ on one side and $E(H)$ on the other and such that, for all $v \in V(H)$ and $e \in E(H)$, there is an edge connecting v and e in $G(H)$ if, and only if, $v \in e$ in H . We say a hypergraph is Berge-acyclic if its incidence graph has no cycles. From now on we shall refer to Berge-acyclic hypergraphs simply as *acyclic hypergraphs*.

(2.5) Definition. A *butterfly* is a connected acyclic uniform hypergraph. The *order* of a finite butterfly is its number of edges.

Fix $l \in \mathbb{N}$.

(2.6) Definition. Let a hypergraph satisfy D_l if the complement of a connected component of maximal size is a disjoint union of finite butterflies, all of them of order less than l .

So D_1 is the event that the hypergraph is a union of a component and some isolated points. For convenience we adopt the convention that D_0 is the event of being connected.

In the case $d = 1$ of graphs, it is a well known result of Erdős and Rényi that the property D_0 of being connected, in spite of not being an elementary one, is asymptotically equivalent to the absence of isolated vertices, obviously an elementary property. The events D_l are generalizations of D_0 to other values of l and d . As one should naturally expect, these generalizations give rise to concepts that are still non-elementary. We shall see that, in analogy with Erdős and Rényi's result, the events D_l are also asymptotically elementary.

The proof that $D_l \notin \mathcal{FO}$ exemplifies a nice use of Compactness.

(2.7) Proposition. For all $l \in \mathbb{N}$, $D_l \notin \mathcal{FO}$.

Proof. Fix l and suppose, for a contradiction, that $D_l \in \mathcal{FO}$. We use compactness.

Let $\{A, B\}$ be a cut in a finite hypergraph. The *norm* of $\{A, B\}$ is the number $\min\{|A|, |B|\}$. Call a cut *bad* if none of the two sides of the cut is a disjoint union of butterflies of order less than l . So D_l is the event that there are no bad cuts.

For each $m \in \mathbb{N}$, let E_m be the event that all bad cuts have norm at least m . By explicitly enumerating and excluding all bad cuts of order $< m$, one sees that $E_m \in \mathcal{FO}$.

Consider the theory $T = \{E_0, E_1, E_2, \dots\} \cup \{\neg D_l\}$. As there are hypergraphs that are a disjoint union of two large butterflies, one sees at once that every finite sub-theory of T is consistent. On the other hand, it is obvious that T is itself inconsistent, in contradiction with compactness.

□

An analysis of the above argument shows that, although D_l is not elementary, it is the class of hypergraphs satisfying all properties in a first order theory, namely the theory $T = \{E_0, E_1, \dots\}$. We say a property P is *axiomatizable* if there is a first order theory T such that, for all hypergraphs H , one has $H \models P$ if, and only if $H \models \sigma$ for all $\sigma \in T$. Of course, if there is such a finite T , P is elementary. So D_l is axiomatizable but not elementary.

Some properties are beyond even the expressible power of first order theories. Further insight on the proof of the above proposition shows that that is the case of the negations $\neg D_l$.

(2.8) Proposition. *For all $l \in \mathbb{N}$, $\neg D_l$ is not axiomatizable.*

Proof. As a contradiction, suppose the theory T axiomatizes D_l . Then the theory

$$T \cup \{E_0, E_1, \dots\}$$

is inconsistent. But, as we have seen above, every finite subtheory of $T \cup \{E_0, E_1, \dots\}$ is consistent, in contradiction with compactness. \square

Still, as the reader can easily verify, if P is *any* property, then the property of *being finite and satisfy P* is axiomatizable.

2.4

Zero-One Laws and Complete Theories

The above observations will be useful in obtaining the following convergence results involving all properties in \mathcal{FO} .

(2.9) Definition. We say a function $p : \mathbb{N} \rightarrow [0, 1]$ is a *zero-one law* if, for all $P \in \mathcal{FO}$, one has

$$\lim_{n \rightarrow \infty} \mathbb{P}(P)(n, p) \in \{0, 1\}.$$

Above we mean that for every $P \in \mathcal{FO}$ the limit exists and is either zero or one.

There is a close connection between zero-one laws and the concept of completeness. We say a theory \mathcal{C} is *complete* if, for every $P \in \mathcal{FO}$, exactly one of $\mathcal{C} \models P$ or $\mathcal{C} \models \neg P$ holds.

Given $p : \mathbb{N} \rightarrow [0, 1]$, the *almost sure theory* of p is defined by

$$T_p := \{P \in \mathcal{FO} \mid \mathbb{P}(P)(n, p) \rightarrow 1\}.$$

So T_p is the class of almost sure elementary properties of $G^{d+1}(n, p)$. Note that, as a contradiction never holds, T_p is always consistent. Moreover as, for every $m \in \mathbb{N}$, the property of having at least m vertices is elementary and holds almost surely, T_p has no finite models.

The connection between completeness and zero-one laws is given by the following.

(2.10) Theorem. *The function p is a zero-one law if, and only if, T_p is complete.*

Proof. Suppose T_p is complete and fix $P \in \mathcal{FO}$. As T_p is complete, either P or $\neg P$ is a semantic consequence of T_p . If $T_p \models P$, by compactness, there is a finite set $\{P_1, P_2, \dots, P_k\} \subseteq T_p$ such that $\{P_1, P_2, \dots, P_k\} \models P$. Therefore

$$\mathbb{P}(P_1 \wedge P_2 \cdots \wedge P_k) \leq \mathbb{P}(P).$$

As $\mathbb{P}(P_1 \wedge P_2 \cdots \wedge P_k) \rightarrow 1$ we have also $\mathbb{P}(P) \rightarrow 1$. Similarly, if $T_p \models \neg P$ one has $\mathbb{P}(\neg P) \rightarrow 1$, so that $\mathbb{P}(P) \rightarrow 0$. As P was arbitrary, p is a zero-one law.

Conversely, if p is a zero-one law then, for any $P \in \mathcal{FO}$, we have either $P \in T_p$ or $\neg P \in T_p$. One cannot have both, as T_p is consistent. So T_p is complete. □

As T_p is consistent and has no finite models, Gödel's Completeness Theorem and Löwenheim-Skolem give that the requirement of T_p being complete is equivalent to asking that all countable models of T_p satisfy exactly the same first-order properties, a situation described in Logic by saying that all countable models are *elementarily equivalent*. One obvious sufficient condition is that T_p be \aleph_0 -categorical, that is, that T_p has, apart from isomorphism, a unique countable model. We shall see several examples where T_p is \aleph_0 -categorical and other examples where the countable models of T_p are elementarily equivalent but not necessarily isomorphic.

We summarize the above observations in the following corollary, more suitable for our applications.

(2.11) Corollary. *A function p is a zero-one law if, and only if, all models of the almost sure theory T_p are elementarily equivalent. In particular, if T_p is \aleph_0 -categorical, then p is a zero-one law.*

Proof. It suffices to show that T_p is complete if, and only if, all countable models of T_p are elementarily equivalent.

Suppose T_p is complete and let A be an elementary property. Then either $T_p \models A$ or $T_p \models \neg A$. In the former case, all countable models of T_p satisfy A , and, in the latter, all such models satisfy $\neg A$. As A is arbitrary, all countable models of T_p are elementarily equivalent.

Now suppose all countable models of T_p are elementarily equivalent and let A be an elementary property. Then either all countable models of T_p satisfy A or all such models satisfy $\neg A$. Suppose the former. We claim that, in this case, $T_p \models A$. For, if that were not the case, the theory $T_p \cup \{\neg A\}$ would be consistent and, by the Downward Löwenheim-Skolem Theorem, would have a

countable model, a contradiction. Therefore, we must have $T_p \models A$. Changing the roles of A and $\neg A$ gives the latter.

For the “in particular”, note that if T_p is \aleph_0 -categorical, all its countable models are elementarily equivalent.

□

Uses of the above result require the ability to recognize when any two models H_1 and H_2 of T_p are elementarily equivalent. This is, usually, a simple matter in case H_1 and H_2 are isomorphic. It is convenient to have at hand an instrument suitable to detecting when two structures of a first-order theory are elementarily equivalent regardless of being isomorphic.

Next, we briefly discuss the definition and some results on the Ehrenfeucht Game, which is a classic example of such an instrument.

2.5

The Ehrenfeucht Game

This game has two players, called Spoiler and Duplicator, and two uniform hypergraphs H_1 and H_2 conventionally on disjoint sets of vertices. These hypergraphs are known to both players. The game has a certain number k of rounds which is again known to both players.

In each round, Spoiler selects one vertex not previously selected in either hypergraph and then Duplicator selects another vertex not previously selected in the other hypergraph. At the end of the k -th round, the vertices x_1, \dots, x_k have been chosen on H_1 and y_1, \dots, y_k on H_2 . Duplicator wins if, for all $\{i_0, i_1, \dots, i_d\} \subseteq \{1, 2, \dots, k\}$, $\{x_{i_0}, \dots, x_{i_d}\}$ is an edge in H_1 if and only if the corresponding $\{y_{i_0}, \dots, y_{i_d}\}$ is an edge in H_2 . Spoiler wins if Duplicator does not. We denote the above described game by $\text{EHF}(H_1, H_2; k)$.

As a technical point, the above description of the game works only if $k \leq \min\{|H_1|, |H_2|\}$. If that is not the case, we adopt the convention that Duplicator wins the game if, and only if, H_1 and H_2 are isomorphic.

The connection of the Ehrenfeucht Game to first order logic is a classic in logic and was given for the first time by R. Fraïssé in his PhD thesis in the more general context of purely relational structures with finite predicate symbols. A proof in the particular case of graphs can be found in Joel Spencer’s book *The Strange Logic of Random Graphs* [2], whose argument applies, *mutatis mutandis* to uniform hypergraphs.

(2.12) Proposition. *The hypergraphs H_1 and H_2 are elementarily equivalent if, and only if, for all $k \in \mathbb{N}$, Duplicator has a winning strategy for the game $\text{EHR}(H_1, H_2; k)$.*

Now it is easy to see the connection of the game to zero-one laws.

(2.13) Corollary. *If for all countable models H_1 and H_2 of T_p and all $k \in \mathbb{N}$ Duplicator has a winning strategy for $EHR(H_1, H_2; k)$ then p is a zero-one law.*

When T_p is \aleph_0 -categorical, the isomorphisms between its countable models give a winning strategy for Duplicator. There are other situations when describing a winning strategy is not that easy.

Consider, for example, with $d = 1$, an infinite tree T_1 all of whose vertices have 3 neighbors. Let also $T_2 = T_1 \sqcup T_1$ be a disjoint union of two copies of T_1 . Then T_1 and T_2 are not isomorphic, but one could naturally guess that Duplicator has a winning strategy for $EHR(T_1, T_2; k)$ for any k : Spoiler will pick two vertices in T_2 and try to show that they lie in different connected components while the corresponding vertices Duplicator chose in T_1 lie in the same. But Duplicator can make it an impossible task by choosing his two vertices suitably far from each other. The next section will turn this idea, and some convenient generalizations, into a precise strategy for Duplicator.

2.6

Some winning strategies for Duplicator

Now we describe some situations when there is a winning strategy for Duplicator without H_1 and H_2 being necessarily isomorphic. All propositions we state below are analogous to propositions in Spencer's book *The Strange Logic of Random Graphs*. There, all results are stated and proved for graphs but, again, all arguments apply, *mutatis mutandis*, to the case of uniform hypergraphs.

In what follows, if H is a $(d + 1)$ -uniform hypergraph and $x \in H$, the *a-neighborhood* of x is the restriction of H to the set of vertices at distance at most a from x . If $x_1, \dots, x_u \in H$, the *a-picture* of x_1, \dots, x_u is the union of the *a-neighborhoods* of the x_i . Let $x_1, \dots, x_u \in H_1$ and $y_1, \dots, y_u \in H_2$. Their *a-pictures* are called *the same* if there is an isomorphism between the *a-pictures* that sends x_i to y_i for all $i \in \{1, \dots, u\}$. Also, $\rho(x, y)$ is the distance from x to y .

(2.14) Proposition. *Set $a = \frac{3^k - 1}{2}$. Suppose H_1 and H_2 have vertex subsets $S_1 \subseteq H_1$ and $S_2 \subseteq H_2$ with the following properties:*

- (a) *The restrictions of H_1 to S_1 and H_2 to S_2 are isomorphic and this isomorphism can be extended to one between the *a-neighborhoods* of S_1 and S_2 .*

- (b) Let $a' \leq a$. Let $y \in H_2$ with $\rho(y, s_2) > 2a' + 1$ for all $s_2 \in S_2$. Let $x_1, \dots, x_{k-1} \in H_1$. Then there is an $x \in H_1$ with x, y having the same a' -neighborhoods and such that $\rho(x, x_i) > 2a' + 1$ for all $1 \leq i \leq k - 1$ and $\rho(x, s_1) > 2a' + 1$ for all $s_1 \in S_1$.
- (c) Let $a' \leq a$. Let $x \in H_1$ with $\rho(x, s_1) > 2a' + 1$ for all $s_1 \in S_1$. Let $y_1, \dots, y_{k-1} \in H_2$. Then there is an $y \in H_2$ with x, y having the same a' -neighborhoods and such that $\rho(y, y_i) > 2a' + 1$ for all $1 \leq i \leq k - 1$ and $\rho(y, s_2) > 2a' + 1$ for all $s_2 \in S_2$.

Then Duplicator has a winning strategy for $EHR(H_1, H_2; k)$.

Proof. Set $a_0 = 0$ and $a_{k+1} = 3a_k + 1$ so that $a_k = \frac{3^k - 1}{2} = a$.

First Duplicator imagines all vertices in $S_1 \cup S_2$ to be marked. Then she will, with s round remaining, try to make the a_s pictures of the marked vertices the same in the two hypergraphs. If she succeeds, at the end the 0-pictures will be the same and she has won. But is that possible? By induction, we see the answer is affirmative:

For $s = k$, that is trivially possible and for $s = k - 1$, the hypothesis guarantee that the set of pictures of individual vertices is the same in both hypergraphs.

Now suppose Duplicator could hold to her strategy until $s = k - j$, that x_1, \dots, x_j have been marked in H_1 and that corresponding y_1, \dots, y_j have been marked in H_2 . For convenience, set $A := a_{s-1}$ and $A' := a_s = 3A + 1$. Then the induction hypotheses is that the A' -pictures of x_1, \dots, x_j and y_1, \dots, y_j are the same.

By symmetry, there is no loss of generality in supposing Spoiler plays a $y_{j+1} \in H_2$. We distinguish two cases:

(Inside) We say Spoiler played *inside* if y_{j+1} is at distance at most $2A + 1$ of some previous y_i . If that is the case, the A -picture of y_1, \dots, y_j, y_{j+1} is contained in the A' -picture of y_1, \dots, y_j and, therefore, is isomorphic to the A -picture of x_1, \dots, x_j . Duplicator then selects a x_{j+1} according to that isomorphism.

(Outside) We say spoiler played *outside* if y_{j+1} is at distance at least $2A + 1$ of any previous y_i . If that is the case, the A -picture of y_{j+1} is a separate component in the A -picture of y_1, \dots, y_j, y_{j+1} . By condition 2, Duplicator can play a x_{j+1} also in a separate component of the new picture.

□

The fact that the Game has a finite number of rounds implies that the pictures described above do not really have to be the same for the conclusion to hold.

The *Distance k -round Ehrenfeucht Game* $\text{DEHR}(H_1, H_2; k)$ on hypergraphs H_1 and H_2 is the game $\text{EHR}(H_1, H_2; k)$ with the additional requirement that for Duplicator to win, she must assure that the distances between corresponding marked vertices are preserved.

We say the a -neighborhoods of x and y are *k -similar* if Duplicator has a winning strategy for the Distance Ehrenfeucht Game in those neighborhoods that begins with x and y marked and has k additional rounds.

Let $x_1, \dots, x_u \in H_1$ and $y_1, \dots, y_u \in H_2$ and call these the marked vertices. The a -picture of x_1, \dots, x_u splits into connected components C_1, \dots, C_r as does the a -picture of y_1, \dots, y_u into $D_1, \dots, D_{r'}$. Suppose that $r = r'$ and that under a suitable renumbering, C_i and D_i contain corresponding marked vertices.

(2.15) Definition. We say that the above a -pictures are *s -similar* if, in addition to the above conditions, for all pairs of components C_i and D_i , duplicator has a winning strategy for the Distance Ehrenfeucht Game that begins with the $x_i \in C_i$ and $y_i \in D_i$ marked and has s additional rounds.

Now we can give a powerful extension of the above result.

(2.16) Proposition. Set $a = \frac{3^k - 1}{2}$. Suppose H_1 and H_2 have vertex subsets $S_1 \subseteq H_1$ and $S_2 \subseteq H_2$ with the following properties:

- (a) S_1 and S_2 have k -similar a -neighborhoods.
- (b) Let $a' \leq a$. Let $y \in H_2$ with $\rho(y, s_2) > 2a' + 1$ for all $s_2 \in S_2$. Let $x_1, \dots, x_{k-1} \in H_1$. Then there is an $x \in H_1$ with x, y having k -similar a' -neighborhoods and such that $\rho(x, x_i) > 2a' + 1$ for all $1 \leq i \leq k - 1$ and $\rho(x, s_1) > 2a' + 1$ for all $s_1 \in S_1$.
- (c) Let $a' \leq a$. Let $x \in H_1$ with $\rho(x, s_1) > 2a' + 1$ for all $s_1 \in S_1$. Let $y_1, \dots, y_{k-1} \in H_2$. Then there is an $y \in H_2$ with x, y having k -similar a' -neighborhoods and such that $\rho(y, y_i) > 2a' + 1$ for all $1 \leq i \leq k - 1$ and $\rho(y, s_2) > 2a' + 1$ for all $s_2 \in S_2$.

Then Duplicator has a winning strategy for $\text{EHR}(H_1, H_2; k)$.

Proof. The strategy is analogous to that of Proposition 14. □

2.7

Convergence Laws

Sometimes a zero-one law is too much to ask, so we present a related weaker concept.

(2.17) Definition. A function $p : \mathbb{N} \rightarrow [0, 1]$ is a *convergence law* if for all $P \in \mathcal{FO}$ the sequence

$$\mathbb{P}(P)(n, p)$$

converges to a real number in $[0, 1]$.

The convergence laws we will deal with occur when the almost sure theory T_p is not far from being complete, in the following sense.

Let T be a first order theory of $(d+1)$ -uniform hypergraphs and suppose we have a collection $\mathcal{C} = \{\sigma_1, \sigma_2, \dots\}$ of first order properties (which is, of course, at most countable). We say \mathcal{C} is a *complete set of completions* (relative to T) if the following conditions hold:

- (a) $T \cup \{\sigma_i\}$ is complete for all i .
- (b) For all $i \neq j$, $T \models \neg(\sigma_i \wedge \sigma_j)$
- (c) For all i , the limit $p_i := \lim_{n \rightarrow \infty} \mathbb{P}(\sigma_i)$ exists.
- (d) $\sum_{i=1}^{\infty} p_i = 1$.

In that case, if H is a hypergraph, and $H \models T$, then exactly one of the following possibilities hold:

$$\begin{aligned} H \models T \cup \{\neg\sigma_1, \neg\sigma_2, \dots\}; \\ H \models T \cup \{\sigma_1\}; \quad H \models T \cup \{\sigma_2\}; \quad H \models T \cup \{\sigma_3\}; \quad \dots \end{aligned}$$

Then item 4 above means that the property axiomatizable by the theory $T \cup \{\neg\sigma_1, \neg\sigma_2, \dots\}$, although not necessarily contradictory, holds almost never.

In case $\mathcal{C} = \{\sigma_1, \sigma_2, \dots\}$ is a complete set of completions for T and A is an elementary property, let $S(A)$ denote the set of indexes i such that $T \cup \{\sigma_i\} \models A$.

(2.18) Proposition. *Under the above conditions, $\lim_{n \rightarrow \infty} \mathbb{P}(A)$ exists for all first order properties A and is given by*

$$\lim_{n \rightarrow \infty} \mathbb{P}(A) = \sum_{i \in S(A)} p_i.$$

We again refer the reader to Spencer's book [2] for a proof.

The next proposition summarizes the above discussion in a way suitable for proving all the convergence laws we need.

(2.19) Proposition. *If T_p admits a complete set of completions then p is a convergence law.*

3

Zero-One Laws for Dense Hypergraphs

The results and arguments in this chapter are essentially those one can find in Spencer's book *The Strange Logic of Random Graphs*, with minor adaptations to fit the case of general edge size $d + 1$.

In the literature of random structures, Corollary 24 is well known: it was proved by Y.V. Glebskii, D.I. Kogan, M.I. Liagonkii and V.A. Talanov in 1969 in [6] (and independently in 1976 by R. Fagin in [7]) in the broader context of purely relational structures.

We get zero-one laws for edge functions that are, in a certain sense, “big”. As a particular case, all constant functions p will be seen to be zero-one laws.

The strategy exemplifies a procedure that will be used many more times: explicit examples of almost sure properties will be given. Then the countable models of these properties will be shown elementarily equivalent, giving the zero-one law. In the present instance the result is stronger: the almost sure theories are \aleph_0 -categorical.

Recall that, for a set S and a natural number k , $\binom{S}{k}$ denotes the collection of all subsets of S of cardinality k .

Fix $r \in \mathbb{N}$ and a set $S \subseteq \binom{\{1,2,\dots,r\}}{d}$. The (r, S) -extension property is the elementary property $E(r, S)$ of $(d + 1)$ -uniform hypergraphs whose interpretation is

“For all x_1, x_2, \dots, x_r , there exists $z \notin \{x_1, x_2, \dots, x_r\}$ such that

$$\sigma(z, x_{i_1}, x_{i_2}, \dots, x_{i_d}) \Leftrightarrow \{i_1, i_2, \dots, i_d\} \in S”.$$

We call such a z a *witness*.

For “big” edge functions, all extension statements, with r fixed, hold almost surely.

(3.1) Proposition. *Let $p \gg n^{-1/\binom{r}{d}}(\log n)^{1/\binom{r}{d}}$ and $1 - p \gg n^{-1/\binom{r}{d}}(\log n)^{1/\binom{r}{d}}$. Then, for all $S \subseteq \binom{\{1,2,\dots,r\}}{d}$, the extension statement $E(r, S)$ holds almost surely.*

Proof. Fix vertices $x_1 \dots x_r$, $S \subseteq \binom{\{1,2,\dots,r\}}{d}$ and $z \notin \{x_1, x_2 \dots x_r\}$.

Suppose first that $p \geq 1/2$. The probability ϵ that z is a witness for S satisfies

$$\epsilon = p^{|S|}(1-p)^{\binom{r}{d}-|S|} \gg n^{-1} \ln n.$$

The case $p \leq 1/2$ is similar. The probability that there is no such z is

$$(1-\epsilon)^{n-r} \sim e^{-n\epsilon} \ll n^{-r}.$$

As there are only $O(n^{-r})$ possibilities for the choices of $\{x_1, x_2, \dots, x_r\}$ and S , the expected number of configurations without witnesses is $o(1)$. By the first moment method, almost surely all configurations have at least one witness and we are done. \square

(3.2) Corollary. *Suppose $p \gg n^{-\epsilon}$ and $1-p \gg n^{-\epsilon}$ for all positive ϵ . Then, for all $r \in \mathbb{N}$ and $S \subseteq \binom{\{1,2,\dots,r\}}{d}$ we have $E(r, S) \in T_p$.*

In particular, the theory $T = \{E(r, S) \mid r \in \mathbb{N}, S \subseteq \binom{\{1,2,\dots,r\}}{d}\}$ is consistent. To get zero-one laws for the above functions p , it is sufficient to show that T is \aleph_0 -categorical.

(3.3) Proposition. *The theory $T = \{E(r, S) \mid r \in \mathbb{N}, S \subseteq \binom{\{1,2,\dots,r\}}{d}\}$ is \aleph_0 -categorical.*

Proof. Let H_1 and H_2 be countable models of T . Let $\{x_1, x_2, \dots\}$ and $\{y_1, y_2, \dots\}$ be enumerations of the vertices of H_1 and H_2 respectively.

Proceeding by induction, we define bijections $a, b : \mathbb{N} \rightarrow \mathbb{N}$ such that the function $F(x_{a(i)}) = y_{b(i)}$ is an isomorphism between H_1 and H_2 .

Begin putting $a(1) = 1$ and $b(1) = 1$ and suppose defined $a(1), \dots, a(r)$ and $b(1), \dots, b(r)$. We distinguish two cases:

If r is even, take $a(r+1) \in \mathbb{N}$ such that $x_{a(r+1)}$ is the element of

$$H_1 \setminus \{x_{a(1)}, x_{a(2)}, \dots, x_{a(r)}\}$$

with smallest index. We take the corresponding $b(r+1)$ such that $y_{b(r+1)}$ is the witness with smallest index that mimics, over $\{y_{a(1)}, \dots, y_{a(r)}\}$, all the incidences of $x_{a(r+1)}$ over $\{x_{a(1)}, \dots, x_{a(r)}\}$.

The case r odd is similar, interchanging the roles of H_1, H_2 and a, b .

Each step uses only vertices not previously mentioned, so a and b are injective. Also, 1 out of every two steps on each side uses the vertex with *smallest index* not previously mentioned, so a and b are surjective.

By construction, the restriction of the correspondence $x_{a(i)} \mapsto y_{b(i)}$ to $\{x_{a(1)}, x_{a(2)}, \dots, x_{a(r)}\}$ is an isomorphism, and that makes F an isomorphism.

□

By the above we have the following.

(3.4) Theorem. *If p is a function such that $p \gg n^{-\epsilon}$ and $1 - p \gg n^{-\epsilon}$ for all positive ϵ , then p is a zero-one law, that is, every elementary property holds almost surely or almost never. Moreover, the choice of almost surely or almost never is the same for all such p .*

Proof. By Corollary 21 and Proposition 22, the theory T is complete and $T \subseteq T_p$, so that T_p is also complete.

As the theory T is the same for every choice of p , we have the “moreover” part.

□

The following particular case is worth mentioning.

(3.5) Corollary. *All constant functions $p \in [0, 1]$ are zero-one laws.*

Proof. The cases $p \in (0, 1)$ are covered by the above result. If $p = 0$ then almost surely the hypergraph has no edges. If $p = 1$, almost surely the hypergraph is complete. In any case, the almost sure theory is \aleph_0 -categorical, so we are done.

□

Next we get to smaller edge functions p that are zero-one laws.

4 Big-Bang

4.1 Counting of Butterfly Components

Now we proceed to investigate zero-one and convergence laws in the early stages of the evolution of $G^{d+1}(n, p)$. More precisely, we investigate edge functions before the double jump:

$$0 \leq p(n) \ll n^{-d}.$$

For functions p in that range, $G^{d+1}(n, p)$ almost surely has no cycles.

(4.1) Proposition. *If $0 \leq p \ll n^{-d}$ then a.a.s. $G^{d+1}(n, p)$ is acyclic. More precisely: if C is a fixed finite cycle, then a.a.s. $G^{d+1}(n, p)$ does not have a copy of C as a sub-hypergraph.*

Proof. Fix a cycle C with v vertices and l edges. Then $v \leq ld$. Consider the expected number \mathbb{E} of copies of H in $G^{d+1}(n, p)$. Then

$$\mathbb{E} \sim O(n^v p^l) = o(1)$$

by the upper bound on p . By the first moment method, a.a.s. there are no copies of C .

□

In view of the above, as far as all our present discussions are concerned, the hypergraphs we deal with are disjoint unions of butterflies. Getting more precise information on the statistics of the number of connected components that are finite butterflies of a given order is the most important piece of information to getting zero-one and convergence laws for $p \ll n^{-d}$.

To this end, we define the following random variables. Below $\delta \in \Delta$, where Δ is the set of all isomorphism classes of butterflies of order l on $v = 1 + ld$ labelled vertices.

(4.2) Definition. $A^\delta(l)$ is the number of finite butterflies of order l and isomorphism class δ in $G^{d+1}(n, p)$.

A butterfly of order l is, in particular, a hypergraph on $v = 1 + ld$ vertices. Let $c^\delta(l)$ be the number of butterflies of order l and isomorphism class δ on $v = 1 + ld$ labelled vertices. To each set S of v vertices on $G^{d+1}(n, p)$ there corresponds the collection of indicator random variables $X_S^1, X_S^2, \dots, X_S^{c^\delta(l)}$, each indicating that one of the potential $c^\delta(l)$ butterflies of order l and isomorphism class δ in S is present and is a component. Therefore one has

$$A(l) = \sum_{S,i} X_S^i,$$

where S ranges over all v -sets and i ranges over $\{1, 2, \dots, c(l)\}$.

Note that a connected component isomorphic to a butterfly of a certain isomorphism class is, in particular, an induced copy of that butterfly. Next we show that the threshold for containment of a butterfly of given order as a connected component is the same for containing butterflies of that order as sub-hypergraphs, not necessarily induced.

In the next proposition, the reader may find the condition

$$p \leq C(\log n)n^{-d}$$

in 2 rather strange, since it mentions functions outside the scope $p \ll n^{-d}$ of the present chapter. The option to putting this more general proposition here reflects the convenience that it has exactly the same proof and that the full condition will be used in the next chapter.

(4.3) Proposition. *Set $v = 1 + ld$. The function $n^{-\frac{v}{l}}$ is a threshold for containment of butterflies of order l as components. More precisely:*

- (a) *If $p \ll n^{-\frac{v}{l}}$ then a.a.s. $G^{d+1}(n, p)$ has no butterflies of order l as sub-hypergraphs.*
- (b) *If $n^{-\frac{v}{l}} \ll p \leq C(\log n)n^{-d}$ where $C < \frac{d!}{1+ld}$ then, for any $k \in \mathbb{N}$ and $\delta \in \Delta$, a.a.s. $G^{d+1}(n, p)$ has at least k connected components isomorphic the butterfly of order l and isomorphism class δ .*

Proof. Let $\mathbb{E}[A^\delta(l)]$ be the expected value of $A^\delta(l)$. One has $\binom{n}{v} \sim \frac{n^v}{v!}$ choices of a set of v vertices, $c^\delta(l)$ choices of the butterfly on it, probability p^l that the l vertices of the butterfly exist and probability $\sim (1-p)^{\binom{n}{d}} \sim \exp[-pv\frac{n^d}{d!}]$ that no other edge connects the butterfly to other components. Therefore

$$\mathbb{E}[A^\delta(l)] \sim c^\delta(l) \frac{n^v}{v!} p^l \exp[-pv\frac{n^d}{d!}].$$

If $p \ll n^{-\frac{v}{l}}$ then $\mathbb{E}[A^\delta(l)] = o(1)$ and, by the first moment method, we have 1.

For 2, suppose $n^{-\frac{v}{i}} \ll p \leq C(\log n)n^{-d}$ where $C < \frac{d!}{1+d}$, so that

$$\mathbb{E}[A^\delta(l)] \rightarrow \infty.$$

Let $\mathbb{V}[A^\delta(l)]$ be the variance of $A^\delta(l)$. It suffices to show that

$$\mathbb{V}[A^\delta(l)] = o(\mathbb{E}[A^\delta(l)]^2).$$

Indeed, by the second moment method, the above condition implies that almost surely $A^\delta(l)$ is close to its expectation $\mathbb{E}[A^\delta(l)] \rightarrow \infty$.

As $\mathbb{V}[A^\delta(l)] = \mathbb{E}[A^\delta(l)^2] - \mathbb{E}[A^\delta(l)]^2$ and $\mathbb{E}[A^\delta(l)] \rightarrow \infty$ the above condition is equivalent to

$$\mathbb{E}[A^\delta(l)^2] \sim \mathbb{E}[A^\delta(l)]^2.$$

We have

$$\begin{aligned} \mathbb{E}[A^\delta(l)^2] &= \mathbb{E} \left[\left(\sum X_S^i \right)^2 \right] \\ &= \mathbb{E} \left[\sum_{S \cap T = \emptyset, i, j} X_S^i X_T^j \right] + \mathbb{E} \left[\sum_{S \cap T \neq \emptyset, i, j} X_S^i X_T^j \right]. \end{aligned}$$

As each X_S^i indicates the presence of a butterfly as an isolated component, the second term is zero. But the first term is

$$\sim (c^\delta(l))^2 \frac{n^{2v}}{v!^2} p^{2l} \exp[-2pv \frac{n^d}{d!}] \sim \mathbb{E}[A^\delta(l)]^2,$$

so we are done. □

Further insight on Proposition 27 gives the following.

(4.4) Theorem. *If $0 \leq p \ll n^{-(d+1)}$ or there is $l \in \mathbb{N}$ such that $n^{-\frac{1+ld}{i}} \ll p \ll n^{-\frac{1+(l+1)d}{l+1}}$ then p is a zero-one law.*

Proof. If $0 \leq p \ll n^{-(d+1)}$, then almost surely there are no edges. As the absence of edges is an elementary property, a model of the almost sure theory in that case is the empty hypergraph in a countable number of vertices. So T_p is \aleph_0 -categorical and all edge functions in that range are zero-one laws.

For $n^{-(1+d)} \ll p \ll n^{-\frac{1+2d}{2}}$, the countable models of the almost sure theory have infinite isolated vertices and infinite isolated edges. This makes T_p \aleph_0 -categorical so all such p 's are zero-one laws.

More generally, for $n^{-\frac{1+ld}{l}} \ll p \ll n^{-\frac{1+(l+1)d}{l+1}}$, the countable models of T_p consist of countably many copies of all butterflies of all orders $\leq l$ and all isomorphism classes and nothing else. That makes T_p \aleph_0 -categorical so these p 's are zero-one laws. □

Let T_l be the first order theory consisting of a scheme of axioms excluding the existence of cycles and butterflies of order $\geq l+1$ and a scheme that assures the existence of infinite copies of each type of butterflies of order $\leq l$. Then T_l is an axiomatization for T_p , where p is as above.

4.2

Just Before the Double Jump

Consider now an edge function p such that for all $\epsilon > 0$, $n^{-(d+\epsilon)} \ll p \ll n^{-d}$ (such functions would include, for instance, $p(n) = (\log n)^{-1}n^{-d}$). The countable models of the almost sure theories for such p 's must be acyclic and have infinite components isomorphic to butterflies of all orders. But in this range a new phenomenon occurs: the existence of components that are butterflies of infinite order is left open. There may or there may not be such components, and therefore the countable models of T_p are not \aleph_0 -categorical.

We proceed to show that these infinite components do not matter from a first-order perspective, as they will be “simulated” by sufficiently large finite components. Because first-order properties are represented by finite formulae, with finitely many quantifications, this will establish that all models of T_p are elementarily equivalent in spite of not being \aleph_0 -categorical.

4.2.1

Rooted Butterflies

The results we state in this section for rooted butterflies are stated and proved in Spencer's book *The Strange Logic of Random Graphs* for rooted trees, which are particular cases of butterflies when $d = 1$. The situation is similar to that of the last section: the same arguments in that book apply to the other values of d .

A *rooted butterfly* is simply a butterfly T (finite or infinite) with one distinguished vertex $R \in T$, called the *root*. With rooted butterflies, the concepts of *parent*, *child*, *ancestor* and *descendent* are clear: their meaning is similar to their natural computer science counterparts for rooted trees. The *depth* of a vertex is its distance from the root. For each $w \in T$, T^w denotes the sub-butterfly consisting of w and all its descendants.

For $r, s \in \mathbb{N}$, we define the (r, s) -value of T by induction on r . Roughly speaking, we examine the r -neighborhood of R and consider any count greater than s , including infinite, indistinguishable from each other and call them “many”. Indeed, the possible $(1, s)$ -values for a rooted tree T are $1, 2, \dots, s, M$ where M stands for “many”. The $(1, s)$ -value of T is then the number of edges incident on the root R if this number is $\leq s$. Otherwise, the $(1, s)$ -value of T is M .

Now suppose the concept of (r, s) -value has been defined for all rooted butterflies and denote by $\text{VAL}(r, s)$ the set of all possible such values. Consider an edge $E = \{R, w_1, \dots, w_d\}$ of T incident on the root R . The *pattern* of E is the function $P : \text{VAL}(r, s) \rightarrow \{1, 2, \dots, d\}$ such that, for all values $\Omega \in \text{VAL}(r, s)$, there are exactly $P(\Omega)$ elements in the set $\{T^{w_1}, \dots, T^{w_d}\}$ with (r, s) -value Ω . Note that

$$\sum_{\Omega \in \text{VAL}(r, s)} P(\Omega) = d.$$

Let $\text{PAT}(r, s)$ be the set

$$\left\{ P : \text{VAL}(r, s) \rightarrow \{1, \dots, d\} \mid \sum_{\Omega \in \text{VAL}(r, s)} P(\Omega) = d \right\}.$$

In other words, $\text{PAT}(r, s)$ is the set of all patterns.

The $(r + 1, s)$ -value of T is the function $V : \text{PAT}(r, s) \rightarrow \{1, 2, \dots, s, M\}$ such that, for all $\Gamma \in \text{PAT}(r, s)$, the root R has exactly $V(\Gamma)$ edges incident on it with pattern Γ , with M standing for “many”.

Note that for any value $\Omega \in \text{VAL}(r, s)$ one can easily create a finite rooted butterfly with value Ω : We simply interpret “many” as $s + 1$. Also, any rooted butterfly can be considered a uniform hypergraph by removing the special designation of the root.

(4.5) Proposition. *Let T_1 and T_2 be rooted butterflies with roots R_1 and R_2 respectively which have the same $(r, s - 1)$ -value. Then, considering T_1 and T_2 as graphs, R_1 and R_2 have (sd) -similar r -neighborhoods.*

Proposition $\tilde{\text{A}}.2.16$ gives the following, which can be interpreted as the formal counterpart of the claim that sufficiently large finite butterflies simulate the behavior of infinite ones as far as elementary properties are concerned.

(4.6) Proposition. *Let H_1 and H_2 be two acyclic graphs in which every finite butterfly occurs as a component an infinite number of times. Then H_1 and H_2 are elementarily equivalent.*

It is convenient to emphasize that, above, H_1 and H_2 may or may not have infinite components.

(4.7) Theorem. *Suppose p is an edge function satisfying, for all $\epsilon > 0$,*

$$n^{-(d+\epsilon)} \ll p \ll n^{-d}.$$

Then p is a zero-one law.

Proof. Consider an edge function p such that $n^{-(d+\epsilon)} \ll p \ll n^{-d}$ for all $\epsilon > 0$. We see that all countable models of T_p satisfy the hypotheses of the above proposition. Therefore they are elementarily equivalent and these p 's are zero-one laws. □

Let T be the first order theory consisting of a scheme of axioms excluding the existence of cycles and a scheme that assures that every finite butterfly of any order appears as a component an infinite number of times. Then T is an axiomatization for T_p .

4.3

On the Thresholds

So far, we have seen that if p satisfies one of the following conditions

- (a) $0 \leq p \leq n^{-(d+1)}$
- (b) $n^{-\frac{1+ld}{l}} \ll p \ll n^{-\frac{1+(l+1)d}{l+1}}$, for some $l \in \mathbb{N}$
- (c) $n^{-(d+\epsilon)} \ll p \ll n^{-d}$ for all $\epsilon > 0$

then p is a zero-one law.

An L -function p in the range $0 \leq p \ll n^{-d}$ that violates all the above three conditions must satisfy, for some $l \in \mathbb{N}$ and $c \in (0, +\infty)$, the condition

$$p(n) \sim c \cdot n^{-\frac{1+ld}{l}}.$$

Informally speaking, in that range, an L -function that is not “between” the thresholds is “on” some threshold. In that case, p is not a zero-one law. Our next goal is to show that those p 's are still convergence laws

4.3.1

Limiting Probabilities on the Thresholds

Let $l \in \mathbb{N}$ and let T_1, T_2, \dots, T_u denote the collection of all possible (up to isomorphism) butterflies of order l and let I be the set of all u -tuples $\mathbf{m} = (m_1, \dots, m_u)$ of non-negative integers. Finally, let $\sigma_{\mathbf{m}}$ be the elementary property that there are precisely m_i components T_i for $i \in \{1, \dots, u\}$.

(4.8) Proposition. *Let $p \sim c \cdot n^{-\frac{1+ld}{l}}$. Then the collection $\{\sigma_{\mathbf{m}} | \mathbf{m} \in I\}$ is a complete set of completions for T_p . In particular, p is a convergence law.*

Proof. We show properties 1, 2, 3 and 4 in the definition of a complete set of completions.

The countable models of $T_p \cup \{\sigma_{\mathbf{m}}\}$ have no cycles, a countably infinite number of components of each butterfly of order $\leq l - 1$, no sub-hypergraph isomorphic to a butterfly of order $\geq l + 1$ and exactly m_i components T_i for each i . So $T_p \cup \{\sigma_{\mathbf{m}}\}$ is \aleph_0 -categorical and, in particular, complete, so we have property 1.

Tautologically no two of the $\sigma_{\mathbf{m}}$ can hold simultaneously, so we have property 2.

For each $i \in \{1, 2, \dots, u\}$, let δ_i be the isomorphism type of T_i . For notational convenience, set $c_i := c^{\delta_i(l)}$ and $A_i := A^{\delta_i(l)}$. The next lemma implies properties 3 and 4 and, therefore, completes the proof. □

(4.9) Lemma. *In the conditions of the above proposition, the random variables A_1, A_2, \dots, A_u are asymptotically independent Poisson with means $\lambda_1 = \frac{c_1}{v!} c^l$, $\lambda_2 = \frac{c_2}{v!} c^l$, \dots , $\lambda_u = \frac{c_u}{v!} c^l$. That is to say,*

$$p_{\mathbf{m}} := \lim_{n \rightarrow \infty} \mathbb{P}(\sigma_{\mathbf{m}}) = \prod_{i=1}^u e^{-\lambda_i} \frac{\lambda_i^{m_i}}{m_i!}.$$

In particular

$$\sum_{\mathbf{m} \in I} p_{\mathbf{m}} = 1.$$

Proof. By the method of factorial moments, it suffices to show that, for all $r_1, r_2, \dots, r_u \in \mathbb{N}$ we have

$$\mathbb{E}[(A_1)_{r_1} \cdots (A_u)_{r_u}] \rightarrow \lambda^{r_1} \cdots \lambda^{r_u}.$$

As we have seen, each A_i can be written as a sum of indicator random variables $A_i = \sum_{S,j} X_S^{i,j}$, each $X_S^{i,j}$ indicates the event $E_S^{i,j}$ that the j -th of the

potential butterflies on the vertex set S is present and is a component. Then

$$\mathbb{E}[(A_1)_{r_1} \cdots (A_u)_{r_u}] = \sum_{S_1, \dots, S_u, j_1, \dots, j_u} \mathbb{P}[E_{S_1}^{1, j_1} \wedge \cdots \wedge E_{S_u}^{u, j_u}].$$

The above sum splits into $\sum_1 + \sum_2$ where \sum_1 consists of the terms with S_1, \dots, S_u pairwise disjoint. As each $X_S^{i, j}$ indicates the presence of a butterfly as a component, we have $\sum_2 = 0$. On the other hand, it is easy to see that if $p \sim c \cdot n^{-\frac{1+ld}{t}}$ then

$$\sum_1 \sim \prod_i c_i^{r_i} \frac{n^{r_i v}}{v!} p^{r_i l} \exp[-pr_i v n^d / d!] \sim \prod_i \lambda^{r_i}, \text{ so we are done.}$$

□

It is worth noting that if a_i is the number of automorphisms of the butterfly whose isomorphism type is δ_i then one has $\frac{c_i}{v!} = \frac{1}{a_i}$.

The convergence laws we got so far provide a nice description of the component structure in the early history of $G^{d+1}(n, p)$: it begins empty, then isolated edges appear, then all butterflies of order two, then all of order three, and so on until $\ll n^{-d}$, immediately before the double jump takes place.

In what follows, BB stands for “Big-Bang”.

(4.10) Definition. BB is the set of all L -functions $p : \mathbb{N} \rightarrow [0, 1]$ satisfying $0 \leq p \ll n^{-d}$.

Now it is just a matter of putting pieces together to get the following.

(4.11) Theorem. All elements of BB are convergence laws.

Proof. Just note that any L -function on the above range satisfies one of the following conditions:

- (a) $0 \leq p \leq n^{-(d+1)}$
- (b) $n^{-\frac{1+ld}{t}} \ll p \ll n^{-\frac{1+(l+1)d}{l+1}}$, for some $l \in \mathbb{N}$
- (c) $n^{-(d+\epsilon)} \ll p \ll n^{-d}$ for all $\epsilon > 0$
- (d) $p \sim c \cdot n^{-\frac{1+ld}{t}}$ for some constant $c \in (0, +\infty)$

□

It is worth noting that the arguments used in getting zero-one laws for the intervals

- (a) $0 \leq p \leq n^{-(d+1)}$
- (b) $n^{-\frac{1+ld}{t}} \ll p \ll n^{-\frac{1+(l+1)d}{l+1}}$, for some $l \in \mathbb{N}$

$$(c) \quad n^{-(d+\epsilon)} \ll p \ll n^{-d} \text{ for all } \epsilon > 0$$

do not require the edge functions to be in Hardy's class, so *all* functions inside those intervals are zero-one laws, regardless of being L -functions.

On the other hand, taking $p = c(n) \cdot n^{-\frac{1+d}{t}}$, where $c(n)$ oscillates infinitely often between two different positive values is sufficient to rule out a convergence law for that edge function.

Also, our discussion implies that, in a certain sense, most of the functions in BB are zero-one laws: the only way one of that functions can avoid this condition is being inside one of the countable windows inside a threshold of appearance of butterflies of some order.

In the next chapter, similar pieces of reasoning will yield an analogous result for another interval of edge functions.

5 Big-Crunch

The present chapter is devoted to getting a result analogous to Theorem 35 on another interval of edge functions, immediately after the double jump. We call that interval BC, for Big-Crunch, because, informally, when “time” (the edge functions p) flows forth, the behavior of the complement of the giant component is the same of the behavior $G^{d+1}(n, p)$ assumes in the Big-Bang BB with time flowing backwards.

More concretely, BC is the set of L -functions p satisfying $n^{-d} \ll p \ll n^{-d+\epsilon}$ for all $\epsilon > 0$. An important function inside this interval is $p = (\log n)n^{-d}$ which will be seen, in the next chapter, to be the threshold for $G^{d+1}(n, p)$ to be connected. In the subintervals $n^{-d} \ll p \ll (\log n)n^{-d}$ and $(\log n)n^{-d} \ll p \ll n^{-d+\epsilon}$, nothing interesting happens in the first order perspective. This will imply that these intervals are entirely made of zero-one laws.

Inside the window $p \sim C \cdot (\log n)n^{-d}$ (with C some positive constant), very much the opposite is true: here we find an infinite collection of thresholds of elementary properties and also an infinite collection of zero-one and convergence laws.

5.1 Just Past the Double Jump

Consider the countable models of the almost sure theory T_p with

$$n^{-d} \ll p \ll (\log n)n^{-d}.$$

As we have already seen, in that range we still have components isomorphic to all finite butterflies of all orders and the possibility of infinite butterflies is still open. The threshold for the appearance of small sub-hypergraphs excludes the possibility of bicyclic (or more) components. By the same reason, we have components with cycles of all types. The following shows, in particular, that vertices of small degree do not occur near the cycles.

(5.1) Proposition. *Suppose $p \gg n^{-d}$. Let H be a finite connected configuration with at least one cycle and at least one vertex of small degree. Then*

the expected number of such configurations in $G^{d+1}(n, p)$ is $o(1)$. In particular a.a.s. there are no such configurations.

Proof. Let the configuration H have v vertices and l edges. As H is connected and has at least one cycle, we have $v \leq ld$. For convenience, set $\alpha = \frac{pm^d}{d!}$. Note that $\alpha \rightarrow +\infty$. Let \mathbb{E} be the expected number of configurations H . Then

$$\begin{aligned} \mathbb{E} &= O\left(\frac{n^v}{v!} p^l (1-p)^{\frac{n^d}{d!}}\right) = O\left(\frac{n^v}{v!} p^l \exp\left[-p \frac{n^d}{d!}\right]\right) = \\ &= O\left(n^{dl} p^l \exp\left[-p \frac{n^d}{d!}\right]\right) t = O\left(\alpha^l \exp[-\alpha]\right) = o(1). \end{aligned}$$

The “in particular” part follows from the first moment method. □

Now it is easy to see that the edge functions in the present range are zero-one laws.

(5.2) Theorem. *Suppose p is an edge function satisfying*

$$n^{-d} \ll p \ll (\log n)n^{-d}.$$

Then p is a zero-one law.

Proof. By Proposition 36, every vertex in the union of all the unicyclic components has infinite neighbors. This determines these components up to isomorphism and it does not pay for Spoiler to play there. But in the complement of the above set, we have already seen that Duplicator can win all k -round Ehrenfeucht Games. Therefore all countable models of T_p are elementarily equivalent and these p are zero-one laws. □

We note that the non-existence of vertices of small degree near cycles is first-order axiomatizable. For each $l, s, k \in \mathbb{N}$ there is a first order sentence which excludes all of the (finitely many) configurations with cycles of order $\leq l$ at distance $\leq s$ from one vertex of degree $\leq k$. Similar considerations show that the non-existence of bicyclic (or more) components is also first-order axiomatizable. So one easily gets a simple axiomatization for the almost sure theories of the above edge functions.

5.2 Beyond Connectivity

Now we consider countable models of T_p with

$$(\log n)n^{-d} \ll p \ll n^{-d+\epsilon}$$

for all positive ϵ .

Again, the thresholds for appearance of small sub-hypergraphs imply that, in this range, we have all cycles of all types as sub-hypergraphs, and no bicyclic (or more) components. Now all vertices of small degree are gone.

(5.3) Proposition. *For $p \gg (\log n)n^{-d}$, the expected number of vertices of small degree in $G^{d+1}(n, p)$ is $o(1)$. In particular, a.a.s. there are no vertices of small degree.*

Proof. Fix a natural number k and let \mathbb{E} be the expected number of vertices of degree k in $G^{d+1}(n, p)$. Then

$$\mathbb{E} \sim n(1-p)^{\frac{n^d}{d!}} \sim n \exp\left[-p \frac{n^d}{d!}\right] = o(1).$$

□

(5.4) Theorem. *Let p be an edge function satisfying*

$$(\log n)n^{-d} \ll p \ll n^{-d+\epsilon}$$

Then p is a zero-one law.

Proof. The countable models of T_p have components that contain cycles of all types, no bicyclic (or more) components and may possibly have butterfly components. As no vertex can have small degree, all vertices in that components have infinite neighbors, so these components are unique up to isomorphism. But T_p is not \aleph_0 -categorical since the existence of butterfly components is left open. Proposition \tilde{A} .2.16 gives that these models are elementarily equivalent, so these p are zero-one laws.

□

The discussion on the proof of theorem \tilde{A} .5.4 also gives simple axiomatizations for the almost sure theories of the above p .

At this point, it is convenient to note a curious fact. We will see, in the next chapter, that if $p \gg (\log n)n^{-d}$ then $G^{d+1}(n, p)$ is almost surely connected. But the countable models of the almost sure theory T_p , described above, are obviously not connected. The solution to this paradox is the fact that, as we

have already seen, connectivity is not an elementary property. Indeed, this paradox can be seen as another way of proving this fact.

5.3

Marked Butterflies

Now we are left to the case of L -functions p comparable to $n^{-d} \log n$. In other words, to complete our discussion, we must get a description of what happens when an edge function p is such that $n^d p / \log n$ tends to a finite constant $C \neq 0$.

In the last chapter, the counting of the connected components isomorphic to butterflies was the fundamental piece of information in the arguments that implied all the convergence laws we found there. Copies of butterflies as connected components are, in particular, induced such copies.

It turns out that the combinatorial structure whose counting is fundamental to getting the convergence laws in the window $p \sim C \cdot \frac{\log n}{n^d}$ is still that of butterflies, but now the copies are not necessarily induced. Instead, some vertices receive markings, meaning that those vertices must have no further neighbors than those indicated on the “model” butterfly. On the non-marked vertices no such requirement is imposed: they are free to bear further neighbors. These copies of butterflies are then, in a sense, “partially induced”.

(5.5) Definition. Let $v^*, l \in \mathbb{N}$. A v^* -marked l -butterfly is a finite connected (Berge)-acyclic hypergraph with l edges and with v^* distinguished vertices, called the *marked* vertices.

Note that a v^* -marked l -butterfly is a hypergraph on $v = 1 + ld$ vertices.

(5.6) Definition. Let B be a v^* -marked l -butterfly and H be a hypergraph. A *copy* of B in H is a (not necessarily induced) sub-hypergraph of H isomorphic to B where if w is a marked vertex of B and w' is the corresponding vertex of H under the above isomorphism, then w and w' have the same degree.

An edge of a Berge-acyclic hypergraph incident to exactly one other edge is called a *leaf*.

(5.7) Definition. A v^* -marked l -butterfly is called *minimal* if every leaf has at least one marked vertex.

Now, the most important concept to understanding the zero-one and convergence laws on BC is the counting of minimal marked butterflies.

(5.8) Definition. Let Γ be the finite set of all isomorphism types of minimal v^* -marked l -butterflies on $1 + ld$ labelled vertices and fix $\gamma \in \Gamma$. Then $c(l, v^*, \gamma)$

is the number of possible v^* -marked l -butterflies of isomorphism type γ on $1+ld$ labelled vertices.

(5.9) Definition. The random variable $A(l, v^*, \gamma)$ is the number of copies of v^* -marked l -butterflies of isomorphism type γ in $G^{d+1}(n, p)$.

5.3.1

Counting of Marked Butterflies

Now we use the first and second moment methods to get precise information on the counting of minimal marked butterflies for edge functions on the range

$$p \sim C \cdot \frac{\log n}{n^d}, \quad C > 0.$$

Rather informally, when the coefficient of $\frac{\log n}{n^d}$ in p avoids the rational value $\frac{d!}{v^*}$ then the expected number of v^* -marked butterflies is either 0 or ∞ . The first moment method implies that, in the first case, a.a.s. there are no v^* -marked butterflies. The second moment method will yield that, in the second case, there are many such minimal marked butterflies.

If $C = \frac{d!}{v^*}$, then knowledge of more subtle behavior of the edge function is required: we are led to consider the coefficient ω of $\frac{\log \log n}{n^d}$ in p . If this coefficient avoids the integer value l then the expected number of v^* -marked l -butterflies is either 0 or ∞ . Again, first and second moment arguments imply that, in the first case, the number of such butterflies is a.a.s. zero and, in the second case, the number of such minimal butterflies is very large.

Finally, if $\omega = l$, then knowledge of even more subtle behavior of the edge function is required: we consider the coefficient c of $\frac{1}{n^d}$ in p . If this coefficient diverges, then the expected number of v^* -marked l -butterflies is either 0 or ∞ and, again, first and second moment methods imply that the actual number of such butterflies is what one expects it to be.

All above cases give rise to zero-one laws. The remaining case is the one when the coefficient c converges. In this case, the fact that the almost sure theories are almost complete will yield convergence laws.

First Moment Analysis of Marked Butterflies

Let $p = p(n)$ be comparable to $\frac{\log n}{n^d}$. That is, let $\frac{n^d p}{\log n}$ converge to a constant $C \neq 0$.

(5.10) Proposition. Fix $\gamma \in \Gamma$.

- (a) If $C < \frac{d!}{v^*}$ then $\mathbb{E}[A(l, v^*, \gamma)] \rightarrow +\infty$.

(b) If $C > \frac{d!}{v^*}$ then $\mathbb{E}[A(l, v^*, \gamma)] \rightarrow 0$ for all $l \in \mathbb{N}$.

In particular, if $C > \frac{d!}{v^*}$ then, for any $l \in \mathbb{N}$, a.a.s. $A(l, v^*, \gamma) = 0$.

Proof. Set $c = c(l, v^*, \gamma)$ and $v = 1 + ld$.

Note that $pv^* \frac{n^d}{d!} \sim Cv^* \frac{\log n}{d!}$ so $pv^* \frac{n^d}{d!} - Cv^* \frac{\log n}{d!} = o(1) \log n$. Therefore one has

$$\begin{aligned} \mathbb{E}[A(l, v^*, \gamma)] &\sim c \frac{n^v}{v!} p^l (1-p)^{v^* \frac{n^d}{d!}} \\ &\sim c \frac{n^v}{v!} p^l \exp \left[-pv^* \frac{n^d}{d!} \right] \\ &\sim c \frac{n^v}{v!} p^l \exp \left[o(1) \log n - Cv^* \frac{\log n}{d!} \right] \\ &\sim c \frac{n^v}{v!} (C \log n)^l n^{-ld} \exp \left[o(1) \log n - Cv^* \frac{\log n}{d!} \right] \\ &\sim \frac{c}{v!} (C \log n)^l n^{1 - \frac{Cv^*}{d!} + o(1)}, \end{aligned}$$

and the result follows.

The ‘‘in particular’’ part follows from the first moment method. \square

Now consider $p \sim \frac{d!}{v^*} \cdot \frac{\log n}{n^d}$ so that $v^* n^d \frac{p}{d!} - \log n = o(1) \log n$ and let

$$\omega(n) = \frac{v^* n^d \frac{p}{d!} - \log n}{\log \log n}.$$

(5.11) Proposition. Fix $l \in \mathbb{N}$ and $\epsilon > 0$.

(a) If eventually $\omega < l - \epsilon$ then $\mathbb{E}[A(l, v^*, \gamma)] \rightarrow +\infty$

(b) If eventually $\omega > l + \epsilon$ then $\mathbb{E}[A(l, v^*, \gamma)] \rightarrow 0$.

In particular, the second condition implies that a.a.s. $A(l, v^*, \gamma) = 0$.

Proof. Set $c := c(l, v^*, \gamma)$ and $v := 1 + ld$.

Note that

$$v^* n^d \frac{p}{d!} = \log n + \omega \log \log n$$

so one has

$$\begin{aligned}
\mathbb{E}[A(l, v^*, \gamma)] &\sim \frac{c}{v!} n^v p^l (1-p)^{v^* \frac{n^d}{d!}} \\
&\sim \frac{c}{v!} n^v p^l \exp\left[-pv^* \frac{n^d}{d!}\right] \\
&\sim \frac{c}{v!} n^v p^l \exp[-\log n - \omega \log \log n] \\
&\sim \frac{c}{v!} n^v \left(\frac{d!}{v^*} \log n\right)^l n^{-ld} n^{-1} (\log n)^{-\omega} \\
&\sim \frac{c}{v!} \left(\frac{d!}{v^*}\right)^l (\log n)^{l-\omega},
\end{aligned}$$

and the result follows.

The “in particular” part follows from the first moment method. \square

Now consider the case $\omega \rightarrow l \in \mathbb{R}$ and let

$$c(n) := p \frac{n^d v^*}{d!} - \log n - l \log \log n.$$

(5.12) Proposition. *Fix $\gamma \in \Gamma$ and $c = c(n)$ as above.*

- (a) *If $c \rightarrow -\infty$ then $\mathbb{E}[A(l, v^*, \gamma)] \rightarrow +\infty$*
- (b) *If $c \rightarrow +\infty$ then $\mathbb{E}[A(l, v^*, \gamma)] \rightarrow 0$.*

In particular, the second condition implies that a.a.s. $A(l, v^, \gamma) = 0$.*

Proof. Note that $p \frac{n^d v^*}{d!} = \log n + l \log \log n + c(n)$, so

$$\begin{aligned}
\mathbb{E}[A(l, v^*, \gamma)] &\sim \frac{c(l, v^*, \gamma)}{v!} n^v p^l (1-p)^{v^* \frac{n^d}{d!}} \\
&\sim \frac{c(l, v^*, \gamma)}{v!} n^v p^l \exp\left[-pv^* \frac{n^d}{d!}\right] \\
&\sim \frac{c(l, v^*, \gamma)}{v!} n^v p^l \exp[-\log n - l \log \log n - c(n)] \\
&\sim \frac{c(l, v^*, \gamma)}{v!} \left(\frac{d!}{v^*}\right)^l \exp[-c(n)],
\end{aligned}$$

and 1 and 2 follow.

The “in particular” part follows from the first moment method. \square

Second moment analysis of marked butterflies

Fix, in $G^{d+1}(n, p)$, a vertex set S of size $|S| = 1 + ld$ and $\gamma \in \Gamma$. To each of the $c := c(l, v^*, \gamma)$ potential copies of v^* -marked l -butterflies of type γ in S there corresponds the random variable X_α , the indicator of the event B_α that this potential copy is indeed there in $G^{d+1}(n, p)$. Then we clearly have

$$A(l, v^*, \gamma) = \sum_{\alpha} X_{\alpha}.$$

We write $|X_\alpha| := S$.

(5.13) Proposition. *Let $p \sim C \cdot \frac{\log n}{n^d}$, where $\frac{d!}{v^{*+1}} < C < \frac{d!}{v^*}$. Then, for any $k, l \in \mathbb{N}$, we have*

$$\mathbb{P}[A(l, v^*, \gamma) \geq k] \rightarrow 1.$$

Proof. By the first moment analysis, the condition on the hypothesis implies $\mathbb{E}[A(l, v^*, \gamma)] \rightarrow +\infty$ and $\mathbb{E}[A(\tilde{l}, \tilde{v}^*, \gamma)] \rightarrow 0$ for all $\tilde{v}^* > v^*$ and any $\tilde{l} \in \mathbb{N}$. We use the second moment method. As

$$\mathbb{E} \left[\sum_{|X_\alpha| \cap |X_\beta| = \emptyset} X_\alpha X_\beta \right] \sim c^2 \frac{n^{2v}}{v!^2} p^{2l} (1-p)^{2v^* \frac{n^d}{d!}} \sim \mathbb{E}[A(l, v^*, \gamma)]^2$$

it suffices to show that

$$\mathbb{E} \left[\sum_{|X_\alpha| \cap |X_\beta| \neq \emptyset} X_\alpha X_\beta \right] = o(1).$$

The sets $|X_\alpha|$ and $|X_\beta|$ can only intersect according to a finite number of patterns, so it suffices to show that the contribution of all terms with a given pattern is $o(1)$. Set $S := |X_\alpha| \cup |X_\beta|$.

Consider an intersection type S such that the model spanned by S contains a cycle. Then the configuration S has a vertex of small degree (marked) near a cycle. The sum of contributions of all terms with that intersection type is \sim the expected number of such configurations. As there is a vertex of small degree near a cycle, this is $o(1)$ by proposition 35.

If the type of S has no cycles, then S is a marked butterfly with $\tilde{v}^* \geq v^*$ marked vertices.

If $\tilde{v}^* > v^*$ then, by the first moment analysis, the sum of contributions of those terms is $o(1)$.

We claim that there are no terms with $\tilde{v}^* = v^*$. Indeed, if that were the case, by minimality, all edges would be in the intersection and so the events indicated by X_α and X_β would be the same, a contradiction. \square

Now consider $p \sim \frac{d!}{v^*} \cdot \frac{\log n}{n^d}$ and, as above, let

$$\omega(n) = \frac{v^* n^{\frac{d}{d!}} - \log n}{\log \log n}.$$

(5.14) Proposition. *Fix $\epsilon > 0$ and $l \in \mathbb{N}$.*

(a) *If eventually $\omega < l - \epsilon$ then for any $k \in \mathbb{N}$, we have*

$$\mathbb{P}[A(l, v^*, \gamma) \geq k] \rightarrow 1$$

(b) *If $\omega \rightarrow +\infty$ then for any $k, \tilde{l} \in \mathbb{N}$, we have*

$$\mathbb{P}[A(\tilde{l}, v^* - 1, \tilde{\gamma}) \geq k] \rightarrow 1$$

for all isomorphism types $\tilde{\gamma}$ of minimal $(v^ - 1)$ -marked \tilde{l} -butterflies.*

Proof. The proof of 1 is the same as the proof of the above proposition.

The proof of 2 is analogous, noting that condition 2 implies that the expected number of v^* -marked butterflies with any fixed number of edges is $o(1)$, so that the intersection pattern must have all the $v^* - 1$ marked vertices. \square

Now consider the case $\omega \rightarrow l$ and, as above, let

$$c(n) = \frac{pn^d v^*}{d!} - \log n - l \log \log n.$$

Finally, the same reasoning used in the proofs of the two above propositions demonstrates the following.

(5.15) Proposition. *If $c \rightarrow -\infty$ then $\mathbb{P}[A(l, v^*, \gamma) \geq k] \rightarrow 1$ for any $k \in \mathbb{N}$.*

5.4

Zero-one laws between the thresholds

Now we consider the countable models of the almost sure theories T_p for p “between” the critical values above.

(5.16) Theorem. *Let p be an edge function satisfying one of the following properties:*

- (a) $p \sim C \cdot \frac{\log n}{n^d}$, where $\frac{d!}{v^*+1} < C < \frac{d!}{v^*}$ for some $d, v^* \in \mathbb{N}$.
- (b) $p \sim \frac{d!}{v^*} \cdot \frac{\log n}{n^d}$ where $\omega \rightarrow \pm\infty$ or $\omega \rightarrow C$ where $l-1 < C < l$ for some $l \in \mathbb{N}$.

Then p is a zero-one law.

Proof. Consider, first, a function $p \sim C \cdot \frac{\log n}{n^d}$, where

$$\frac{d!}{v^*+1} < C < \frac{d!}{v^*}.$$

We consider the models of the almost sure theory $T_{v^*} := T_p$. In that range, we still have no bicyclic (or more) components in the first-order perspective. As there are no vertices of small degree near cycles, the unicyclic components are determined up to isomorphism. Also we still have infinitely many copies of each cycle. So the union of connected components containing cycles are determined up to isomorphism and Duplicator does not have to worry about them: every time Spoiler plays there, he has wasted a move.

So let us consider the butterfly components. By the first and second moment analysis above, we have no components containing (v^*+1) -marked butterflies and have infinite components containing copies of each minimal v^* -marked butterfly. Each component containing a v^* -marked butterfly is determined up to isomorphism: each non-marked vertex must have infinite neighbors.

Let $l \in \mathbb{N}$ such that $1+ld \leq v^* < v^*+1 \leq 1+(l+1)d$. Then there are no butterflies of order $l+1$ (or more) as sub-hypergraphs and there are infinitely many components isomorphic to each butterfly of order $\leq l$. Therefore, the union of the components isomorphic to finite butterflies is determined up to isomorphism.

T_{v^*} is not \aleph_0 -categorical, though, since in that countable models, there may or may not be components containing \tilde{v}^* -marked butterflies with $\tilde{v}^* < v^*$. (This includes the degenerate case $\tilde{v}^* = 0$: there may or may not be infinite butterflies where all vertices have infinite neighbors) These components are “simulated” by components containing v^* -marked vertices, with $v^* - \tilde{v}^*$ marked

vertices suitably far from the neighborhood of the \tilde{v}^* marked vertices, this neighborhood being a copy of the \tilde{v}^* -marked butterfly one wants to simulate.

More precisely, it is clear that the countable models of T_p satisfy the hypothesis of Proposition $\tilde{A}.2.16$, so they are pairwise elementarily equivalent and, hence, T_{v^*} is complete and the corresponding p are zero-one laws.

Now consider $p \sim \frac{d!}{v^*} \cdot \frac{\log n}{n^d}$ and, as above, let

$$\omega(n) = \frac{v^* n^{\frac{d}{d!}} - \log n}{\log \log n}.$$

If $\omega \rightarrow -\infty$ then the first and second moment analysis above imply that the countable models of T_p are the same as the countable models of T_{v^*} and, as T_{v^*} is complete, p is a zero-one law.

If $\omega \rightarrow +\infty$ then the first and second moment analysis above imply that the countable models of T_p are the same as the countable models of T_{v^*-1} . But the latter theory is complete and, hence, the corresponding p are zero-one laws.

If $\omega \rightarrow C$, with $l-1 < C < l$, then the countable models of $T_{v^*}^l := T_p$ are the same as the countable models of T_{v^*} but without the components with marked butterflies of order $\leq l-1$. These models are, for the same reasons, still pairwise elementarily equivalent, so we have that the corresponding p are zero-one laws.

Finally, consider the case $\omega \rightarrow l$ and, as above, let

$$c(n) = \frac{pn^d v^*}{d!} - \log n - l \log \log n.$$

If $c \rightarrow -\infty$, then the analysis above show that the countable models of T_p are the same as the countable models of $T_{v^*}^l$, so these p are zero-one laws.

If $c \rightarrow +\infty$, then the analysis above show that the countable models of T_p are the same as the countable models of $T_{v^*}^{l-1}$, so these p are also zero-one laws.

□

5.5 Axiomatizations

At this point, it is clear that the arguments given in the last section actually give axiomatizations for the almost sure theories presented there.

More formally, let the theory $T(v^*)$ consist of a scheme of axioms saying that there are no bicyclic (or more) components, a scheme of axioms saying that there are no copies of \tilde{v}^* -marked butterflies for each $\tilde{v}^* > v^*$ and a scheme of

axioms saying that there are infinitely many copies of each minimal v^* -marked butterfly.

Similarly, let the theory $T(v^*, l)$ consist of a scheme of axioms saying that there are no bicyclic (or more) components, a scheme of axioms excluding the \tilde{v}^* -marked butterflies for each $\tilde{v}^* > v^*$, a scheme of axioms saying that there are no v^* -marked butterflies of order $\leq l - 1$ and an scheme saying that there are infinitely many copies of each minimal v^* -marked butterfly not excluded by the last scheme.

By the discussion found in the last section, we have the following:

(5.17) Theorem. *The theory $T(v^*)$ is an axiomatization for T_{v^*} and, similarly, the theory $T(v^*, l)$ is an axiomatization for $T_{v^*}^l$.*

5.6

On the thresholds

The only way an L -function can avoid all of the clauses discussed above is the possibility that $c(n)$ converges to a real number c . That is to say, we must consider the possibility that

$$p = \frac{d!}{v^*} \cdot \frac{\log n + l \log \log n + c(n)}{n^d}$$

where $c(n) \rightarrow c$.

We will see, in the present chapter, that these p , although not zero-one laws, are still convergence laws. The situation is analogous to that of the last chapter: on these thresholds, the almost sure theories T_p are almost complete.

5.6.1

Limiting Probabilities on the Thresholds

Let $v^*, l \in \mathbb{N}$ and let T_1, T_2, \dots, T_u denote the collection of all possible (up to isomorphism) v^* -marked butterflies of order l and let I be the set of all u -tuples $\mathbf{m} = (m_1, \dots, m_u)$ of non-negative integers. Finally, let $\sigma_{\mathbf{m}}$ be the elementary property that there are precisely m_i components T_i for $i \in \{1, \dots, u\}$.

(5.18) Proposition. *Let $p = \frac{d!}{v^*} \cdot \frac{\log n + l \log \log n + c(n)}{n^d}$, where $c(n) \rightarrow c$. Then the collection $\{\sigma_{\mathbf{m}} | \mathbf{m} \in I\}$ is a complete set of completions for T_p . In particular, p is a convergence law.*

Proof. We show properties 1, 2, 3 and 4 in the definition of a complete set of completions.

We claim the countable models of $T_p \cup \{\sigma_{\mathbf{m}}\}$ are pairwise elementarily equivalent. Indeed, the complement of the union of components containing

the v^* -marked butterflies of order l is elementarily equivalent to the countable models of the theory $T_{v^*}^{l+1}$, defined above. As the latter theory is complete, T_p is also complete, and we have 1.

Tautologically no two of the σ_m can hold simultaneously, so we have property 2.

For each $i \in \{1, 2, \dots, u\}$, let δ_i be the isomorphism type of T_i . For notational convenience, set $c_i := c(l, v^*, \delta_i)$ and $A_i := A(l, v^*, \delta_i)$. The next lemma implies properties 3 and 4 and, therefore, completes the proof. \square

(5.19) Lemma. *In the conditions of the above proposition, the random variables A_1, A_2, \dots, A_u are asymptotically independent Poisson with means*

$$\lambda_i = \frac{c_i}{v!} \left(\frac{d!}{v^*} \right)^l e^{-c}.$$

That is to say,

$$p_m := \lim_{n \rightarrow \infty} \mathbb{P}(\sigma_m) = \prod_{i=1}^u e^{-\lambda_i} \frac{\lambda_i^{m_i}}{m_i!}.$$

In particular

$$\sum_{m \in I} p_m = 1.$$

Proof. By the method of factorial moments, it suffices to show that, for all $r_1, r_2, \dots, r_u \in \mathbb{N}$ we have

$$\mathbb{E}[(A_1)_{r_1} \cdots (A_u)_{r_u}] \rightarrow \lambda^{r_1} \cdots \lambda^{r_u}.$$

As we have seen, each A_i can be written as a sum of indicator random variables $A_i = \sum_{S,j} X_S^{i,j}$, each $X_S^{i,j}$ indicates the event $E_S^{i,j}$ that the j -th of the potential copies of v^* -marked l -butterflies on the vertex set S is present. Then

$$\mathbb{E}[(A_1)_{r_1} \cdots (A_u)_{r_u}] = \sum_{S_1, \dots, S_u, j_1, \dots, j_u} \mathbb{P}[E_{S_1}^{1,j_1} \wedge \cdots \wedge E_{S_u}^{u,j_u}].$$

The above sum splits into $\sum_1 + \sum_2$ where \sum_1 consists of the terms with S_1, \dots, S_u pairwise disjoint. It is easy to see that if $p = \frac{d!}{v^*} \frac{\log n + l \log \log n + c(n)}{n^d}$ then $\sum_1 \sim \prod_i \lambda^{r_i}$.

Arguing as in Proposition 45, one sees that the contribution of each of the terms in \sum_2 with a given pattern of intersection is $o(1)$. Hence $\sum_2 = o(1)$ and we are done. \square

These pieces together prove the following.

(5.20) Theorem. *All elements in BC are convergence laws.*

Proof. Just note that all L -functions on the above range must satisfy, with the familiar definitions of $\omega(n)$ and $c(n)$, one of the following conditions:

- (a) $n^{-d} \ll p \ll (\log n)n^{-d}$
- (b) $(\log n)n^{-d} \ll p \ll n^{-d+\epsilon}$ for all positive ϵ
- (c) $p \sim C \cdot \frac{\log n}{n^d}$, where $\frac{d!}{v^*+1} < C < \frac{d!}{v^*}$ for some $d, v^* \in \mathbb{N}$.
- (d) $p \sim \frac{d!}{v^*} \cdot \frac{\log n}{n^d}$ where $\omega \rightarrow \pm\infty$ or $\omega \rightarrow C$ where $l-1 < C < l$ for some $l \in \mathbb{N}$
- (e) $\omega \rightarrow l \in \mathbb{N}$ and $c(n) \rightarrow \pm\infty$ or $c(n) \rightarrow c \in \mathbb{R}$.

□

As it was the case in the last chapter, it is worth noting that the arguments used in getting zero-one laws for the clauses 1, 2 and 3 do not require the edge functions to be in Hardy's class, so *all* functions inside those intervals are zero-one laws, regardless of being L -functions.

On the other hand, taking $\omega(n)$ oscillating infinitely often between two constant values $\omega_1 < l$ and $\omega_2 > l$ makes the probability of an elementary event oscillate between zero and one. Similarly, taking $c(n)$ oscillating between any two different positive values makes the probability of an elementary event oscillate between two different values $\notin \{0, 1\}$. Obviously, these situations rule out convergence laws.

As it was the case with BB, our present discussion implies that, in a certain sense, most of the functions in BC are zero-one laws: the only way one of that functions can avoid this condition is being inside one of the countable windows inside a threshold for the presence of marked butterflies of some order.

6 Some elementary approximations

In this chapter we describe some combinatorial aspects of the component structure of the random hypergraph for $p \sim C \frac{\log n}{n^d}$ that will, later, be used to get elementary approximations to the non-elementary events D_l . It all starts by showing that Erdős and Rényi's threshold for connectivity has a nice generalization for random hypergraphs.

6.1 Threshold for Connectivity

Now we show that $p = d! \frac{\log n}{n^d}$ is a threshold for $G^{d+1}(n, p)$ to be connected. The 0-statement we already know: if $p = C \cdot \frac{\log n}{n^d}$ with $C < d!$, then a.a.s. there are many isolated vertices, so $G^{d+1}(n, p)$ is almost never connected. The 1-statement is the following.

(6.1) Theorem. *Let $p(n) = \frac{C \log n}{n^d}$, where $C > d!$. Then a.a.s. $G^{d+1}(n, p)$ is connected.*

Proof. Let $a > 0$ be such that $C > \frac{(d+1)!}{d+1-a^d}$. The expected number of cuts is less than or equal to

$$\sum_{k=1}^{n/2} \binom{n}{k} (1-p)^{\binom{n}{d+1} - \binom{n-k}{d+1} - \binom{k}{d+1}}.$$

This sum is less than or equal to

$$\sum_{k=1}^{an} \binom{n}{k} (1-p)^{\binom{n}{d+1} - \binom{n-k}{d+1} - \binom{k}{d+1}} + \sum_{k=an}^{n/2} \binom{n}{k} (1-p)^{\binom{n}{d+1} - \binom{n-k}{d+1} - \binom{k}{d+1}}$$

so it suffices to show that the two latter sums are $o(1)$.

The first sum is less than or equal to

$$\begin{aligned}
& \sum_{k=1}^{an} n^k \exp \left\{ -\frac{p}{(d+1)!} [n^{d+1} - (n-k)^{d+1} - k^{d+1}] \right\} \\
& \leq \sum_{k=1}^{an} n^k \exp \left\{ -\frac{p}{(d+1)!} [k(d+1)n^d + o(n^d) - k^{d+1}] \right\} \\
& \leq \sum_{k=1}^{an} n^k \exp \left\{ -\frac{pk}{(d+1)!} [(d+1)n^d + o(n^d) - (k-1)^d] \right\} \\
& \leq \sum_{k=1}^{an} n^k \exp \left\{ -\frac{pk}{(d+1)!} [(d+1)n^d + o(n^d) - (an-1)^d] \right\} \\
& \leq \sum_{k=1}^{an} n^k \exp \left\{ -\frac{Ck \log n}{(d+1)!n^d} [(d+1)n^d + o(n^d) - (an-1)^d] \right\} \\
& \leq \sum_{k=1}^{an} n^k \exp \left\{ -\frac{Ck \log n}{(d+1)!} [(d+1) + o(1) - a^d] \right\} \\
& = \sum_{k=1}^{an} \exp \left\{ k \log n - \frac{Ck \log n}{(d+1)!} [(d+1) + o(1) - a^d] \right\} \\
& = \sum_{k=1}^{an} \exp \left\{ k \log n \left[1 - \frac{C}{(d+1)!} ((d+1) + o(1) - a^d) \right] \right\}.
\end{aligned}$$

By the choice of a , there is $M > 0$ such that, for sufficiently large n , the latter sum is less than or equal to

$$\sum_{k=1}^{an} \exp \{-Mk \log n\}.$$

But we have

$$\sum_{k=1}^{an} \exp \{-Mk \log n\} \leq \sum_{k=1}^{an} n^{-Mk} \leq \frac{n^{-M}}{1 - n^{-M}} = o(1).$$

The second sum is less than or equal to

$$\sum_{k=an}^{n/2} 2^n \exp \left\{ -p \left[\binom{n}{d+1} - \binom{n-k}{d+1} - \binom{k}{d+1} \right] \right\}.$$

The log of the summand is

$$\begin{aligned}
& n \log 2 - \frac{C \log n}{(d+1)!n^d} [n^{d+1} - (n-k)^{d+1} - k^{d+1}] \\
& \leq n \log 2 - \frac{Cn \log n}{(d+1)n^d} \left[(n-1)^d - \left(1 - \frac{k}{n}\right) (n-k-1)^d - \frac{k}{n} (k-1)^d \right] \\
& \leq n \log 2 - \frac{Cn \log n}{(d+1)n^d} \left[(n-1)^d - \left(1 - \frac{k}{n}\right) (n-1)^d - \frac{k}{n} (n/2-1)^d \right] \\
& \leq n \log 2 - \frac{Cn \log n}{(d+1)n^d} \left[\frac{k}{n} (n^d + o(n^d)) - \frac{k}{n} \left(\frac{n^d}{2^d} + o(n^d) \right) \right] \\
& \leq n \log 2 - \frac{Cn \log n}{(d+1)} \left[\frac{k}{n} \left(1 - \frac{1}{2^d} \right) + o(1) \right] \\
& \leq n \log 2 - \frac{Cn \log n}{(d+1)} \left[a \left(1 - \frac{1}{2^d} \right) + o(1) \right]
\end{aligned}$$

This is the log of an individual summand. We have at most n terms, so the sum is less than or equal to

$$\exp \left\{ \log n + n \log 2 - \frac{Cn \log n}{(d+1)} \left[a \left(1 - \frac{1}{2^d} \right) + o(1) \right] \right\}$$

and this is $o(1)$ as the $n \log n$ term overwhelms all others.

□

6.2 Component Structure on BC

It is possible to explore the information we have until now and, particularly, the arguments in the last section, to get more precise information on the component structure of $G^{d+1}(n, p)$ for $p \in \text{BC}$.

(6.2) Lemma. *Let E_1 be the event that there is a sub-hypergraph isomorphic to a butterfly with $l+1$ edges and E_2 the event that there is no connected component isomorphic to a butterfly on l edges.*

Then, for $p = p(n) \leq \frac{C \log n}{n^d}$, with $C < \frac{dl}{1+ld}$, the probability of the event $E_1 \wedge E_2$ is $o(1)$.

Proof. Fix a positive real α such that $\frac{1+(l+1)d}{l+1} < \alpha < \frac{1+ld}{l}$ and consider two cases:

If $p \leq n^{-\alpha}$ then a.a.s. there is no sub-hypergraph isomorphic to a butterfly with $l+1$ vertices.

If $p \geq n^{-\alpha}$ then a.a.s. there is a connected component isomorphic to a butterfly with l vertices.

□

(6.3) Definition. The random variable μ is the number of vertices outside the largest connected component of $G^{d+1}(n, p)$.

The last part of the argument in theorem \tilde{A} .6.1 gives the following lemma.

(6.4) Lemma. Let $a > 0$ and $C > 0$. Then for $p(n) = C \frac{\log n}{n^d}$ one has a.a.s. $\mu < an$.

Proof. This argument occurs, with virtually no changes, in the last part of the proof of theorem \tilde{A} .6.1, but we repeat it here for the convenience of the reader.

If $\mu \geq an$ then $G^{d+1}(n, p)$ has a cut with at least an vertices on the smaller side. The expected number of such cuts is less than or equal to

$$\sum_{k=an}^{n/2} \binom{n}{k} (1-p)^{\binom{n}{d+1} - \binom{n-k}{d+1} - \binom{k}{d+1}}$$

which in turn is less than or equal to

$$\sum_{k=an}^{n/2} 2^n \exp \left\{ -p \left[\binom{n}{d+1} - \binom{n-k}{d+1} - \binom{k}{d+1} \right] \right\}.$$

The log of the summand is

$$\begin{aligned} & n \log 2 - \frac{C \log n}{(d+1)! n^d} [n^{d+1} - (n-k)^{d+1} - k^{d+1}] \\ & \leq n \log 2 - \frac{Cn \log n}{(d+1)n^d} \left[(n-1)^d - \left(1 - \frac{k}{n}\right) (n-k-1)^d - \frac{k}{n} (k-1)^d \right] \\ & \leq n \log 2 - \frac{Cn \log n}{(d+1)n^d} \left[(n-1)^d - \left(1 - \frac{k}{n}\right) (n-1)^d - \frac{k}{n} (n/2-1)^d \right] \\ & \leq n \log 2 - \frac{Cn \log n}{(d+1)n^d} \left[\frac{k}{n} (n^d + o(n^d)) - \frac{k}{n} \left(\frac{n^d}{2^d} + o(n^d) \right) \right] \\ & \leq n \log 2 - \frac{Cn \log n}{(d+1)} \left[\frac{k}{n} \left(1 - \frac{1}{2^d} \right) + o(1) \right] \\ & \leq n \log 2 - \frac{Cn \log n}{(d+1)} \left[a \left(1 - \frac{1}{2^d} \right) + o(1) \right] \end{aligned}$$

This is the log of an individual summand. We have at most n terms, so the sum is less than or equal to

$$\exp \left\{ \log n + n \log 2 - \frac{Cn \log n}{(d+1)} \left[a \left(1 - \frac{1}{2^d} \right) + o(1) \right] \right\}$$

and this is $o(1)$ as the $n \log n$ term overwhelms all others.

□

(6.5) Theorem. *Let $p \sim \frac{C \log n}{n^d}$ where C is a constant such that $C > \frac{d!}{1+d}$. Then a.a.s. the random hypergraph $G^{d+1}(n, p)$ has no sub-hypergraph isomorphic to a butterfly on l vertices outside the largest connected component.*

Proof. Fix positive reals $\tilde{C} > C$, $\alpha < \frac{d}{1+(l+1)d}$ and a such that $a\tilde{C} < \frac{d!}{1+d}$.

Let A be the event that there is a sub-hypergraph isomorphic to a butterfly with $l + 1$ edges outside the giant component and B the event that there is a connected component isomorphic to a butterfly on l vertices.

For p in this range, we know that $\mathbb{P}(B) = o(1)$. Therefore, it suffices to show that

$$\mathbb{P}(A) = o(1).$$

Let μ be the number of vertices outside the giant component and define $m_1 = m_1(n)$ to be the natural number $m \in [0, n^\alpha]$ that maximizes

$$\mathbb{P}(A|\mu = m).$$

Similarly, define $m_2 = m_2(n)$ to be the natural number $m \in [n^\alpha, an]$ that maximizes $\mathbb{P}(A \wedge \neg B|\mu = m)$. One has

$$\mathbb{P}(A) = \mathbb{P}(A \wedge B) + \mathbb{P}(A \wedge \neg B) \leq \mathbb{P}(B) + \mathbb{P}(A \wedge \neg B).$$

As $\mathbb{P}(B) = o(1)$ it suffices to show that $\mathbb{P}(A \wedge \neg B) = o(1)$. We have

$$\mathbb{P}(A \wedge \neg B) \leq \mathbb{P}(A \wedge \neg B \wedge \mu \leq n^\alpha) + \mathbb{P}(A \wedge \neg B \wedge n^\alpha \leq \mu \leq an) + \mathbb{P}(\mu \geq an).$$

We show that the three latter terms are $o(1)$.

The third term is $o(1)$ by lemma \tilde{A} .6.4.

For the first one, note that

$$\begin{aligned} \mathbb{P}(A \wedge \neg B \wedge \mu \leq n^\alpha) &\leq \mathbb{P}(A \wedge \mu \leq n^\alpha) \\ &= \sum_{m=0}^{n^\alpha} \mathbb{P}(\mu = m) \mathbb{P}(A|\mu = m) \\ &\leq \mathbb{P}(A|\mu = m_1). \end{aligned}$$

Note that $\mathbb{P}(A|\mu = m_1)$ is the probability that the random hypergraph on m_1 vertices and probability $p(n)$ has a sub-hypergraph isomorphic to a butterfly with $l + 1$ edges, and that this is at most the expected number of

butterflies with $l + 1$ edges. So

$$\mathbb{P}(A|\mu = m_1) = O \left[(\log n)n^{-d}m_1^{1+(l+1)d} \right] = O \left[(\log n)n^{\alpha(1+(l+1)d)-d} \right] = o(1)$$

by the choice of α .

As for the second term, one has

$$\begin{aligned} \mathbb{P}(A \wedge \neg B \wedge \mu \geq n^\alpha) &= \sum_{n^\alpha}^{an} \mathbb{P}(\mu = m)\mathbb{P}(A \wedge \neg B|\mu = m) \\ &\leq \mathbb{P}(A \wedge \neg B|\mu = m_2). \end{aligned}$$

Note that $\mathbb{P}(A \wedge \neg B|\mu = m_2)$ is the probability that the random hypergraph on m_2 vertices and probability $p(n)$ has a sub-hypergraph isomorphic to a butterfly on $l + 1$ vertices and no connected component isomorphic to a butterfly on l vertices.

It is easy to see that the function $n \mapsto \frac{\log n}{n^{d-1}}$ is eventually decreasing, so that $\frac{\log n}{n^{d-1}} \leq \frac{\log m_2}{m_2^{d-1}}$ for sufficiently large n . Moreover, it is obvious that $m_2 < an$.

Putting all together, one has, for sufficiently large n ,

$$p < \frac{\tilde{C} \log n}{n^d} \leq a\tilde{C} \cdot \frac{\log m_2}{m_2^d} < \frac{d!}{1+ld} \cdot \frac{\log m_2}{m_2^d}$$

so that, by Lemma 52 and the fact that $m_2 \rightarrow \infty$, we have

$$\mathbb{P}(A \wedge \neg B|\mu = m_2) = o(1).$$

□

For the reader who feels uneasy about the logic of the above argument, note, for example, that what is actually been done in the last part is the construction of a function

$$m_2 \mapsto p(m_2) \in [0, 1]$$

beginning with the image of the function $m_2(n)$ and completing with values of p that do not contradict the already existing inequalities. The only reason for which the sequence

$$\mathbb{P}(A \wedge \neg B|\mu = m_2(n))$$

fails to be a subsequence of

$$\mathbb{P} [G^{d+1}(m_2, p(m_2)) \models E_1 \wedge E_2]$$

is the fact that it may have repetitions. But, still, it is a *sub-net* of the latter sequence, because $m_2(n) \rightarrow \infty$. As the latter converges to zero, this suffices to getting the desired conclusion $\mathbb{P}(A \wedge \neg B | \mu = m_2(n)) = o(1)$. Similar reasonings are needed in the arguments found in the next section but, due to their cumbersome but trivial nature, we will no longer bother the reader with such explicit formulations.

Putting all we already know about the existence of butterflies as components gives the following theorem, which gives a description of the “disappearance” of the butterfly components as time goes forth in the window $p \sim C \frac{\log n}{n^d}$. The butterflies of larger order are incorporated to the giant component before the butterflies of smaller order, so, outside the giant component, what we see is the behavior of $G^{d+1}(n, p)$ in BB but with time flowing backwards.

In all that follows, let

$$c_l(n) = \frac{n^d(1+ld)}{d!} p(n) - \log n - l \log \log n.$$

Notice that, if $\omega \rightarrow l$, then $c_l(n)$ is the usual $c(n)$.

(6.6) Theorem. *Let p be such that*

$$\lim_{n \rightarrow \infty} c_l(n) = -\infty, \quad \lim_{n \rightarrow \infty} c_{l+1}(n) = +\infty.$$

Fix $k \in \mathbb{N}$.

Then a.a.s. the complement of the largest component of $G^{d+1}(n, p)$ consists of a disjoint union of butterflies of order at most l and nothing else. Moreover, for each isomorphism type of each butterfly of order $\leq l$, there are at least k copies of butterflies of that type as components.

6.2.1

Some Estimates for μ

The arguments in the proof of the above theorem can be used to get upper and lower bounds for μ that are much better than that of Lemma 54.

(6.7) Theorem. *Let $p \sim C \frac{\log n}{n^d}$ satisfy*

$$\lim_{n \rightarrow \infty} c_l(n) = -\infty.$$

Fix any function $f(n) \ll n^{\frac{d}{1+ld}} (\log n)^{-1-ld}$.

Then a.a.s. $\mu > f(n)$.

Proof. Let A be the event that there is a sub-hypergraph isomorphic to a butterfly of order l . Then the above condition on p implies $\mathbb{P}(\neg A) = o(1)$. We have

$$\begin{aligned} \mathbb{P}(\mu \leq f(n)) &= \mathbb{P}(A \wedge \mu \leq f(n)) + \mathbb{P}(\neg A \wedge \mu \leq f(n)) \\ &\leq \mathbb{P}(A \wedge \mu \leq f(n)) + \mathbb{P}(\neg A). \end{aligned}$$

As $\mathbb{P}(\neg A) = o(1)$, it suffices to show that $\mathbb{P}(A \wedge \mu \leq f(n)) = o(1)$. To this end, let $m_1 = m_1(n)$ be the natural number $m \in [0, f(n)]$ that maximizes $\mathbb{P}(A \mid \mu = m)$. Then

$$\mathbb{P}(A \wedge \mu \leq f(n)) = \sum_{m=0}^{f(n)} \mathbb{P}(\mu = m) \mathbb{P}(A \mid \mu = m) \leq \mathbb{P}(A \mid \mu = m_1).$$

Note that $\mathbb{P}(A \mid \mu = m_1)$ is the probability that the random hypergraph on m_1 vertices and edge probability $p(n)$ has a sub-hypergraph isomorphic to a butterfly on l vertices and that this is at most the expected number of such butterflies. Then

$$\mathbb{P}(A \mid \mu = m_1) = O[(\log n)n^{-d}m_1^{1+ld}] = O[(\log n)n^{-d}(f(n))^{1+ld}] = o(1)$$

by the condition on f . □

(6.8) Theorem. *Let p satisfy*

$$\lim_{n \rightarrow \infty} c_l(n) = +\infty.$$

Fix any function $f(n) \gg n^{\frac{ld}{1+ld}}(\log n)^{-\frac{l}{1+ld}}$.

Then a.a.s. $\mu < f(n)$.

Proof. Let A be the event that there is a connected component isomorphic to a butterfly of order l . Then any of the above conditions on p implies that $\mathbb{P}(A) = o(1)$. Therefore, as above, it suffices to show that

$$\mathbb{P}(\neg A \wedge \mu \geq f(n)) = o(1).$$

To this end, fix a and let m_2 be the natural number $m \in [f(n), an]$ that maximizes $\mathbb{P}(\neg A \mid \mu = m)$. Then

$$\mathbb{P}(\neg A \wedge \mu \geq f(n)) = \sum_{m=f(n)}^{an} \mathbb{P}(\mu = m) \mathbb{P}(\neg A \mid \mu = m) \leq \mathbb{P}(\neg A \mid \mu = m_2).$$

Note that $\mathbb{P}(\neg A \mid \mu = m_2)$ is the probability that the random hypergraph on m_2 vertices and edge probability $p(n)$ has no component isomorphic to a butterfly on l vertices. But the conditions on f and a imply, for sufficiently large n ,

$$m_2^{-\frac{1+ld}{l}} \ll p(n) \leq \frac{d!}{1+ld} \cdot \frac{\log m_2}{m_2^d}.$$

Therefore $\mathbb{P}(\neg A \mid \mu = m_2) = o(1)$. □

6.2.2 Some Elementary Approximations

The description of the structure outside the giant component given in the above section enables us to get good elementary approximations to the events D_l . Below we define a class of properties that are, in a sense, asymptotically very improbable.

(6.9) Definition. \mathcal{I} is the set of all properties P of $(d+1)$ -uniform hypergraphs such that $\mathbb{P}(P) \rightarrow 0$ for all L -functions $p : \mathbb{N} \rightarrow [0, 1]$.

Note that \mathcal{I} is an *ideal*, in that if $P_1 \subseteq P_2 \in \mathcal{I}$ then $P_1 \in \mathcal{I}$. Also, \mathcal{I} is non-trivial, since the tautological event \top is not an element of \mathcal{I} .

We say two events P_1 and P_2 are *asymptotically equivalent* if their symmetric difference $P_1 \triangle P_2$ is an element of \mathcal{I} . In that case we write

$$P_1 \equiv P_2 \pmod{\mathcal{I}}.$$

Let A be the property that there are no components isomorphic to butterflies of order l and B the property that the largest component is a butterfly of order l and all other components are butterflies of order $< l$. Then $\tilde{D}_l := A \vee B$ is an elementary property and is a good approximation to the event D_l .

(6.10) Theorem. For all $l \in \mathbb{N}$, $D_l \equiv \tilde{D}_l \pmod{\mathcal{I}}$.

Proof. Note that $D_l \subseteq \tilde{D}_l$, so that $D_l \Delta \tilde{D}_l = \tilde{D}_l \setminus D_l$.

First, if $p \ll n^{-\frac{1+ld}{l}}$, then $\lim_{n \rightarrow \infty} \mathbb{P}(D_l) = \lim_{n \rightarrow \infty} \mathbb{P}(\tilde{D}_l) = 1$.

So suppose, from now on, that $p \gg n^{-\frac{1+ld}{l}}$.

If p satisfies

$$\lim_{n \rightarrow \infty} c_l(n) = +\infty$$

then $\lim_{n \rightarrow \infty} \mathbb{P}(D_l) = \lim_{n \rightarrow \infty} \mathbb{P}(\tilde{D}_l) = 1$.

Now suppose $p \gg n^{-\frac{1+ld}{l}}$ satisfy

$$\lim_{n \rightarrow \infty} c_l(n) = -\infty.$$

Then $\lim_{n \rightarrow \infty} \mathbb{P}(D_l) = \lim_{n \rightarrow \infty} \mathbb{P}(\tilde{D}_l) = 0$.

There are two remaining cases:

(a) $p \sim Cn^{-\frac{1+ld}{l}}$

(b) $p = \frac{d!}{1+ld} \cdot \frac{\log n + l \log \log n + c(n)}{n^d}$, where $c(n) \rightarrow c \in \mathbb{R}$.

In both of them, the almost sure properties we already know to hold in $G^{d+1}(n, p)$ easily imply that $\mathbb{P}(\tilde{D}_l \setminus D_l) \rightarrow 0$.

□

Note that in the non-trivial cases

(a) $p \sim Cn^{-\frac{1+ld}{l}}$

(b) $p = \frac{d!}{1+ld} \cdot \frac{\log n + l \log \log n + c(n)}{n^d}$, $c(n) \rightarrow c \in \mathbb{R}$

Lemmas 4.9 and 5.19 imply, with the notation we find there, that

$$\lim_{n \rightarrow \infty} \mathbb{P}[D_l] = \exp\left[-\sum_{i=0}^u \lambda_i\right]$$

where $v = 1 + ld$ and $\lambda_i = \frac{c_i}{v!} C^l$ in the first case and $\lambda_i = \frac{c_i d!}{v! v} e^{-c}$, in the second, which is a generalization of Erdős beautiful “double exponential” formula

$$\lim_{n \rightarrow \infty} \mathbb{P}[G(n, p) \text{ is connected}] = e^{-e^{-c}}$$

for $p = \frac{\log n + c}{n}$.

The above approximations are global, in the sense that they work for all ranges of L -functions p . There are other situations in which the approximations have a more local character, meaning that they work for some specific ranges of p . Consider, for example, the non-elementary predicate $C(x)$, meaning that x belongs to the largest connected component of $G^{d+1}(n, p)$. Then the

above discussions imply that if p is appropriately large then $C(x)$ is almost surely equivalent to the predicate $\tilde{C}_l(x)$, meaning that x does not belong to a component of order $\leq l$. Obviously, $\tilde{C}_l(x)$ is elementary for all $l \in \mathbb{N}$. Formally,

(6.11) Proposition. *If p satisfies*

$$\lim_{n \rightarrow \infty} c_l(n) = +\infty$$

then the predicate $C(x)$ is almost surely equivalent to the predicate $\tilde{C}_l(x)$.

7

Future Directions

Lynch [8] shows that $p(n) = \lambda/n$ is a convergence law for $G(n, p)$ if $\lambda > 0$ is any constant, so the gap between BB and BC is filled at least in the case $d = 1$. It would be interesting to check whether the same situation holds for the general random hypergraph $G^{d+1}(n, p)$.

This work has considered zero-one and convergence laws for edge functions p on the ranges $p, 1 - p \gg n^{-\epsilon}$ for all $\epsilon > 0$ and $p \ll n^{-d+\epsilon}$ for all $\epsilon > 0$. There are many edge functions outside those ranges and it would be interesting to study the almost sure theories of some of them so as to get information about possible convergence laws. For example, Shelah and Spencer, in [9], showed that if $\alpha \in (0, 1)$ is an irrational number then $p = n^{-\alpha}$ is a zero-one law for the binomial random graph $G(n, p)$. So zero-one laws remain a frequent appearance outside the considered ranges, at least in the case $d = 1$ of random graphs. It is natural to ask whether the functions $n^{-\alpha}$, for $\alpha \in (0, d) \setminus \mathbb{Q}$ are zero-one laws for $G^{d+1}(n, p)$. The methods introduced by Shelah and Spencer seem to apply, with minor modifications in this case, to give a positive answer to that question.

The fact that all the above irrational powers of n are zero-one laws could make one wonder whether it would be possible to improve the exponent on the right hand side of BC = $(n^{-d}, n^{-d+\epsilon})$ while still getting convergence laws. Spencer, in [2], shows that if α is any rational number in $(0, 1)$, then $n^{-\alpha}$ fails to be a convergence law, so the answer is negative at least in the case $d = 1$ of random graphs. Again, the methods Spencer introduces seem to apply to the other values of d to give a negative answer, and, therefore, the exponent in the right hand of BC is probably the best possible. Still, it would be interesting to have a complete description of the intervals of L -functions entirely made of convergence laws.

One could also look for examples of other elementary approximations of non-elementary properties, as it was the case with D_l and \tilde{D}_l . An interesting more challenging project is to describe the asymptotic expressive power of the first order logic of uniform hypergraphs, that is, the class of all properties P such that $P \equiv \tilde{P} \pmod{\mathcal{I}}$ for some elementary property \tilde{P} . For example, if one

could get a nice description of the elements of the ideal \mathcal{I} , then the asymptotic expressive powers of all classes of properties would also be described.

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